

ON ROUQUIER BLOCKS FOR FINITE CLASSICAL GROUPS AT LINEAR PRIMES

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ABSTRACT. H. Miyachi and W. Turner have independently proved that Broué's Abelian Defect Group Conjecture holds for certain unipotent blocks of the finite general linear group, the so-called Rouquier blocks[18] and[19, Section 2, Theorem 1]. This together with A. Marcus[17, Theorem 4.3(b)] and J. Chuang and R. Rouquier[7, Theorem 7.18] proves that the conjecture holds for all blocks of such groups. We prove that other finite classical groups also possess unipotent Rouquier blocks at linear primes.

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1. INTRODUCTION AND NOTATION

Let p be a prime and consider the p -modular system (K, \mathcal{O}, k) such that K contains enough roots of unity for the groups being considered in this paper.

Let G be a finite group and b a block idempotent of $\mathcal{O}G$ with defect group P . Recall that to each subgroup H of G containing $N_G(P)$ there exists a unique block $\mathcal{O}Hc$ associated to b through the Brauer homomorphism.

We state Broué's Abelian Defect Group Conjecture[14, Chapter 6.3.3].

Conjecture 1.0.1 (Broué). *Let G be a finite group and P an abelian p -subgroup. Let b be a block idempotent of $\mathcal{O}G$ with defect group P and Brauer correspondent c in $N_G(P)$. Then $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$ are derived equivalent.*

The conjecture is known to hold for symmetric groups. Recall that to each block B of $\mathcal{O}S_n$ there is an associated non-negative integer w called the weight of B . The defect group P of B is abelian if and only if $w < p$, in which case $P \cong (C_p)^w$. The proof consists of three steps:

- (1) For each w with $(0 \leq w < p)$ there exists an $n \geq 1$ and a block B of $\mathcal{O}S_n$ which is Morita equivalent to the principal block of $\mathcal{O}(S_p \wr S_w)$ [6, Section 3, Theorem 2].
- (2) The principal block of $\mathcal{O}(S_p \wr S_w)$ is derived equivalent to the Brauer correspondent of B in $N_{S_n}(P)$ [17, Theorem 4.3(b)].
- (3) Any two blocks B, B' of $\mathcal{O}S_n, \mathcal{O}S_{n'}$ respectively of the same weight have Morita equivalent Brauer correspondents.
- (4) Any two blocks B, B' of $\mathcal{O}S_n, \mathcal{O}S_{n'}$ respectively of the same weight are derived equivalent[7, Theorem 7.2].

These methods have been adapted for unipotent blocks of finite general linear groups[19, Section 2, Theorem 1], [7, Theorem 7.18]. We investigate (1), (2) and (3) for unipotent blocks of other finite classical groups.

Let q be a prime power (we allow q to be even only in the case of the unitary group) and $G = G_m(q)$ be a group of the form $U_n(q), Sp_{2n}(q), CSp_{2n}(q), SO_{2n+1}(q), SO_{2n}^+(q), SO_{2n}^-(q), CSp_{2n}^+(q)$ or $CSp_{2n}^-(q)$. We adopt the notation that elements of $U_n(q)$ have entries in \mathbb{F}_q so q is a square, say $q = q_0^2$. For a positive integer d we let J_d be the $d \times d$ matrix with entries in \mathbb{F}_q with 1s along the antidiagonal and 0s elsewhere. To each of the above groups types we associate the 2-fold extension $GL_d(q).2 = \langle GL_d(q), s \rangle$ of $GL_d(q)$ described as follows:

(i)

$$s^2 = \begin{cases} -1 & \text{for } Sp_{2n}(q) \text{ or } CSp_{2n}(q) \\ 1 & \text{otherwise} \end{cases}$$

(ii) and for all $A \in GL_d(q)$

$$sAs^{-1} = \begin{cases} J_d A^{-q_0 t} J_d & \text{for } U_n(q) \\ -J_d A^{-t} J_d & \text{for } Sp_{2n}(q) \text{ or } CSp_{2n}(q) \\ J_d A^{-t} J_d & \text{otherwise} \end{cases}$$

We have a natural homomorphism $GL_d(q).2 \rightarrow \{\pm 1\}$ with kernel $GL_d(q)$. This extends to a map $GL_d(q).2 \wr S_w \rightarrow \{\pm 1\}$ and we use M to denote the kernel of this map.

We now recall some facts about the unipotent blocks of $\mathcal{O}G$, more details can be found in section 8. To each unipotent block B of $\mathcal{O}G$ there is an associated non-negative integer w called the weight of B . As with the symmetric group the defect group of B is abelian if and only if $w < p$. In the case of $SO_{2n}^+(q)$ or $CSO_{2n}^+(q)$ we have the notion of B being degenerate. For a fixed prime power q we have the notion of p being a linear or unitary prime with respect to q .

Now let p be a linear prime with respect to q and d the multiplicative order of $q \pmod p$.

We now state our main theorem.

Theorem 1.0.2. (cf. Theorem 9.1.1, Theorem 9.2.1, Corollary 9.3.1, Lemma 9.3.2) *Let p be a linear prime with respect to q . For $CSp_{2n}(q)$ and $CSO_{2n}^\pm(q)$ let's denote by Z_p the p -part of the multiplicative group \mathbb{F}_q^\times . For all other groups Z_p will be the trivial group. For all w ($0 \leq w < p$) there exists some $m = m(w)$ and some unipotent block B of $\mathcal{O}G_m(q)$ of weight w such that:*

- (1) *B is Morita equivalent to the principal block of $\mathcal{O}((GL_d.2 \wr S_w) \times Z_p)$ if $G = U_n(q)$, $Sp_{2n}(q)$, $CSp_{2n}(q)$ or $SO_{2n+1}(q)$ or if $G = SO_{2n}^\pm(q)$ or $CSO_{2n}^\pm(q)$ and B is non-degenerate.*
- (2) *B is Morita equivalent to the principal block of $\mathcal{O}(M \times Z_p)$ if $G = SO_{2n}^+(q)$ or $CSO_{2n}^+(q)$ and B is degenerate.*

We will then prove that the Brauer correspondent of B in $N_G(P)$ is derived equivalent to the appropriate block in the above theorem and thus we obtain:

Corollary 1.0.3. (cf. Corollary 10.0.3) *The block B of $\mathcal{O}G$ satisfies Broué's Abelian Defect Group Conjecture.*

Finally we will prove that if B and B' are unipotent blocks of $\mathcal{O}G_m(q)$ and $\mathcal{O}G_{m'}(q)$ respectively with the same weight and either both non-degenerate or both degenerate then they have Morita equivalent Brauer correspondents (see Corollary 10.0.6). We will then have an analogue for parts (1), (2) and (3) in the proof of the conjecture for the symmetric groups for unipotent blocks at linear primes of our finite classical groups.

We should note that the Morita equivalences we construct in 9.1.1, 9.2.1 and 9.3.1 are splendid in the sense of [14, Chapter 9.2.5] and that they ultimately gives rise to splendid derived equivalences between B and its Brauer correspondent in $N_G(P)$.

Sections 2,3 and 4 give some basic background information before our finite classical groups are introduced. In section 2 we describe the combinatorial objects partitions and symbols. They will later be used to label the characters and blocks of our finite classical groups. In section 3 we will take a brief look at the Weyl groups of type A_n , B_n and D_n including branching rules which will ultimately be used to describe what happens to characters of our finite classical groups under Harish-Chandra induction. Section 4

consists of a brief overview of the Brauer homomorphism and the Brauer correspondence.

Section 5 is where we first introduce the finite classical groups that are the subject of this paper. We do this in two ways. First as the group of fixed points of some algebraic group under some Frobenius endomorphism and secondly as the group of linear maps that preserve or scale some bilinear form. We then go on to describe certain Levi subgroups of the finite classical groups. In section 6 we set up a labeling for the characters of the groups in question. Section 7 describes Harish-Chandra induction from the Levi subgroups described in section 5 and we look at what effect this has on characters. Unipotent blocks are introduced in section 8 and we describe the defect groups as well as exactly what characters are in such a block.

Our main theorems 9.1.1 and 9.2.1 are proved in section 9 and we then go on to look at how the main theorems are applied to Broué's Abelian Defect Group Conjecture. In section 10 we go on to prove the analogues of steps (2) and (3) in the proof of Broué's Abelian Defect Group Conjecture for the symmetric groups.

2. SOME COMBINATORICS

2.1. Partitions and Symbols. For our description of partitions and symbols we follow G. Hiss and R. Kessar[12, Section 2.2].

A partition λ of a non-negative integer n is a finite ordered set $(\alpha_1, \alpha_2, \dots, \alpha_t)$ of positive integers with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t$ and $\alpha_1 + \alpha_2 + \dots + \alpha_t = n$. We write $\lambda \vdash n$ or $|\lambda| = n$. A beta-set of a partition $\lambda = (\alpha_1, \alpha_2, \dots, \alpha_t)$ is a finite set of non-negative integers $\{\beta_1, \beta_2, \dots, \beta_s\}$ where $\beta_1 < \beta_2 < \dots < \beta_s$ and $\alpha_i = \beta_{s-i+1} - s + i$ for $1 \leq i \leq s$ and α_i is considered to be zero for $i > t$.

Now if d is a non-negative integer then the d -shift of a beta-set $\{\beta_1, \beta_2, \dots, \beta_s\}$ is $\{0, 1, \dots, d-1\} \cup \{\beta_1 + d, \beta_2 + d, \dots, \beta_s + d\}$.

Two beta-sets are said to be equivalent if one is the d -shift of the other for some non-negative integer d and two beta-sets give rise to the same partition if and only if they are equivalent.

A symbol is an unordered pair of beta-sets $\{X, Y\}$. A symbol $\{X, Y\}$ is said to be degenerate if $X = Y$. Two symbols are said to be equivalent if one of them can be obtained from the other by simultaneous d -shifts on both parts of the symbol for some non-negative integer d .

The defect of a symbol $\{X, Y\}$ is the quantity:

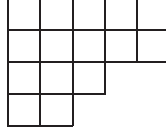
$$||X| - |Y||$$

The rank is:

$$\sum_{x \in X} + \sum_{y \in Y} - \lfloor (\frac{|X|+|Y|-1}{2})^2 \rfloor$$

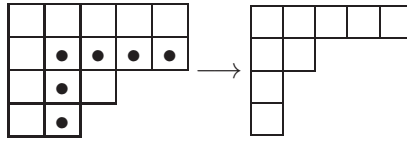
Both defect and rank are constant on equivalence classes of symbols.

2.2. Hooks and Cores. The Young diagram for a partition $\lambda = (\alpha_1, \alpha_2, \dots, \alpha_t)$ is an arrangement of boxes with α_i boxes in the i th row. For example the partition $(5, 5, 3, 2)$ has Young diagram:



A hook of a partition is any box in its Young diagram together with everything directly below it and directly to the right of it in the diagram. The length of a hook is equal to the number of boxes in the hook. If a hook is of length e it is said to be an e -hook.

The partition corresponding to the diagram obtained by deleting a hook and moving everything below and to the right of the hook one box up and one box left is said to be obtained by removing this hook. For example:



shows a 6-hook and how to delete it.

If β is a corresponding beta-set then an e -hook corresponds to a pair of non-negative integers (x, y) such that $x \in \beta$ and $y \notin \beta$ with $x - y = e$. Removing this hook corresponds to replacing x with y in β .

Similarly if β is a beta-set and (x, y) a pair of non-negative integers with $x \in \beta$, $y \notin \beta$ and $y - x = e$ then the beta-set obtained by replacing x with y is said to be obtained by adding an e -hook to β . The corresponding transformation to the partition is also called adding an e -hook.

For the rest of the section let's fix a positive integer e . Beta-sets can be displayed on an e -abacus as follows:

Take an abacus with e columns (runners) labeled from left to right by $0, 1, \dots, e-1$ and with the rows labeled from the top by $0, 1, 2, \dots$. From now on this will be known as an e -abacus. Now given some beta-set β we can represent it on the abacus by putting a bead on the j th runner and i th row if and only if $ei + j \in \beta$.

Removing/adding an e -hook corresponds to moving a bead up/down a runner one place into an unoccupied position. From this description it is clear that removing e -hooks until there are no more e -hooks to remove gives a well-defined beta-set called the e -core of β . The corresponding partition is called the e -core of the partition corresponding to β . Again this is well-defined, in other words equivalent beta-sets have equivalent cores. If λ is a partition and λ' its e -core then the e -weight of λ is defined to be $\frac{|\lambda| - |\lambda'|}{e}$. In other words the weight is defined to be the number of e -hooks removed to obtain the e -core.

Given a beta-set β we can construct a beta-set β^j by letting $i \in \beta^j$ if and only if $ei + j \in \beta$. These beta-sets are called the e -quotients of β . Then a beta-set is completely defined by the β^j s ($0 \leq j \leq e-1$). The e -quotients are written as $[\beta^0, \beta^1, \dots, \beta^{e-1}]$. If λ is the partition corresponding to β then we say $[\lambda^0, \lambda^1, \dots, \lambda^{e-1}]$ are the e -quotients

of λ where λ^i is the partition corresponding to β^i . This is uniquely determined by λ up to a cyclic permutation of the λ^i 's.

Given a symbol $\{X, Y\}$ we again have a notion of hook. An e -hook is a pair (x, y) of non-negative integers with $x \in X$ but $y \notin X$ (or X replaced with Y) with $x - y = e$. Removing this e -hook means replacing x with y in X (or Y). There is also a completely analogous notion of adding hooks.

Now consider the $2e$ -abacus. For integers i and j with $0 \leq i$ and $0 \leq j \leq e - 1$ put a bead on the i th row and j th runner if and only if $ei + j \in X$ and on the i th row and $(e + j)$ th runner if and only if $ei + j \in Y$. This is called the $2e$ -linear diagram of $\{X, Y\}$.

This diagram shows that removing e -hooks until no more can be removed is a well-defined process producing what is called the e -core of $\{X, Y\}$. Also equivalent symbols have equivalent e -cores. If when we remove all e -hooks from a symbol we get a degenerate symbol and this process involved removing a positive number of e -hooks we say that the e -core is 2 copies of this degenerate symbol. However, if this process involved removing no e -hooks then we say the e -core is just 1 copy of this degenerate symbol. From now on we will use symbol to mean an equivalence class of symbols.

2.3. Alternative Description. When we are looking at characters of our finite classical groups we will need the following alternative description of partitions and symbols as given in [12, Section 5.2].

Given a partition we display it on a 2-abacus making sure the 0th runner has more beads than the 1th. Say it has s more beads. Then we relabel the partition (s, μ, ν) where $[\mu, \nu]$ are the 2-quotient partitions with respect to the abacus representation given. Given a symbol $\{X, Y\}$ with $|X| > |Y|$ we relabel it (s, μ, ν) where $s = |X| - |Y|$, μ is the partition corresponding to X and ν is the partition corresponding to Y . If we have a symbol $\{X, Y\}$ with $|X| = |Y|$ we relabel it $(0, \mu, \nu)$ where μ is the partition corresponding to X and ν is the partition corresponding to Y . Note that in this final case $(0, \mu, \nu)$ and $(0, \nu, \mu)$ correspond to the same symbol.

2.4. Littlewood-Richardson Coefficients. Let μ, ν and λ be partitions such that $|\mu| = m$, $|\nu| = n$ and $|\lambda| = (n + m)$. The Littlewood Richardson coefficient $g_{\mu, \nu}^\lambda$ is described as follows [13, Definition 16.1]:

Let $\mu = (\alpha_1, \alpha_2, \dots, \alpha_s)$, $\nu = (\beta_1, \beta_2, \dots, \beta_t)$ and $\lambda = (\gamma_1, \gamma_2, \dots, \gamma_u)$. If $s > u$ or $\alpha_i > \gamma_i$ for some i with $0 \leq i \leq s$ then $g_{\mu, \nu}^\lambda$ is zero.

Otherwise lie the Young diagram of μ on top of that of λ so that the top left boxes coincide. Now $g_{\mu, \nu}^\lambda$ is the number of ways the complement of the Young diagram of μ inside that of λ can be filled with positive integers according to the following rules.

- (1) There must appear β_i i 's.
- (2) The integers must be non-decreasing from left to right along rows and strictly increasing down columns.

- (3) Reading each row from right to left starting with the top row and continuing downwards gives a sequence of integers such that at no point does the number of $(i+1)$ s exceed the number of i s for all positive integers i .

3. WEYL GROUPS

For the branching rules of the following Weyl groups we follow G. Hiss and K. Kessar[12, Section 3]. We will have repeated use of the Littlewood-Richardson coefficients 2.4.

A_n: The Weyl group of type A_n , the symmetric group S_{n+1} on $(n+1)$ letters, has presentation:

$$S_{n+1} = \langle s_1, \dots, s_n \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1 \mid i - j > 1 \rangle$$

The ordinary characters of S_n are labeled by partitions of n . If $\alpha \vdash n$ we use χ^α to denote the corresponding character of S_n .

We are concerned with inducing characters from $S_{n-k} \times S_k$, generated by:

$$\{s_1, \dots, s_{n-k-1}, s_{n-k+1}, \dots, s_{n-1}\}$$

to S_n .

If $\alpha \vdash (n-k)$, $\beta \vdash k$ and $\gamma \vdash n$ then the multiplicity of χ^γ in $\text{Ind}_{S_{n-k} \times S_k}^{S_n}(\chi^\alpha \otimes \chi^\beta)$ is $g_{\alpha, \beta}^\gamma$.

B_n: The Weyl group of type B_n , denoted W_n , has presentation:

$$W_n = \langle s_1, \dots, s_n \mid s_i^2 = 1, (s_1 s_2)^4 = 1, (s_i s_{i+1})^3 = 1 \mid i > 1, (s_i s_j)^2 = 1 \mid i - j > 1 \rangle$$

The ordinary characters of W_n are labeled by bi-partitions of n , a bi-partition of n is an ordered pair of partitions (α^0, α^1) such that $|\alpha^0| + |\alpha^1| = n$ and we write $(\alpha^0, \alpha^1) \vdash n$. If $(\alpha^0, \alpha^1) \vdash n$ we use χ^α to denote the corresponding character of W_n .

We are concerned with inducing characters from $W_{n-k} \times S_k$, generated by:

$$\{s_1, \dots, s_{n-k}, s_{n-k+2}, \dots, s_n\}$$

to W_n .

If $(\alpha^0, \alpha^1) \vdash (n-k)$, $\gamma \vdash k$ and $(\beta^0, \beta^1) \vdash n$ then set $j = |\beta^0| - |\alpha^0|$. The multiplicity of χ^β in $\text{Ind}_{W_{n-k} \times S_k}^{W_n}(\chi^\alpha \otimes \chi^\gamma)$ is zero if $j < 0$ or $j > k$. Otherwise it is:

$$\sum_{\delta^0 \vdash j} \sum_{\delta^1 \vdash k-j} g_{\alpha^0, \delta^0}^{\beta^0} g_{\alpha^1, \delta^1}^{\beta^1} g_{\delta^0, \delta^1}^\gamma$$

D_n: The Weyl group of type D_n , denoted \widetilde{W}_n , has presentation:

$$\widetilde{W}_n = \langle s_1, \dots, s_n \mid s_i^2 = 1, (s_1 s_2)^2 = 1, (s_1 s_3)^3 = 1, (s_i s_{i+1})^3 = 1 \mid i > 1, (s_i s_j)^2 = 1 \mid i - j > 1 \mid \{i, j\} \neq \{1, 3\} \rangle$$

Note \widetilde{W}_n can be viewed as a subgroup of W_n of index 2 with generators $\{s_1 s_2 s_1, s_2, \dots, s_n\}$.

If $(\alpha^0, \alpha^1) \vdash n$ with $\alpha^0 \neq \alpha^1$ then the character $\chi^{(\alpha^0, \alpha^1)}$ of W_n restricts to a single character of \widetilde{W}_n and $\chi^{(\alpha^1, \alpha^0)}$ restricts to the same character. We denote this character $\chi^{\{\alpha^0, \alpha^1\}}$ and describe it as non-degenerate. However, if $\alpha^0 = \alpha^1$ then $\chi^{(\alpha^0, \alpha^0)}$ restricts to the sum of 2 distinct characters of \widetilde{W}_n . We denote the 2 characters $\chi^{\{\alpha^0, \alpha^0\}}$ and $\chi'^{\{\alpha^0, \alpha^0\}}$ and describe them as degenerate. Now conjugation by any element $g \in W_n \setminus \widetilde{W}_n$ induces an automorphism on \widetilde{W}_n that swaps $\chi^{\{\alpha^0, \alpha^0\}}$ and $\chi'^{\{\alpha^0, \alpha^0\}}$.

We are concerned with inducing characters from $\widetilde{W}_{n-k} \times S_k$, generated by:

$$\{s_1, \dots, s_{n-k}, s_{n-k+2}, \dots, s_n\}$$

to \widetilde{W}_n .

Given the information above we can deduce the multiplicity of an irreducible character of \widetilde{W}_n in that of an irreducible character of $\widetilde{W}_{n-k} \times S_k$ induced up to \widetilde{W}_n . We consider three cases. When the characters of \widetilde{W}_{n-k} and \widetilde{W}_n are both non-degenerate, when the character of \widetilde{W}_{n-k} is non-degenerate but that of \widetilde{W}_n is degenerate and when the character of \widetilde{W}_{n-k} is degenerate but that of \widetilde{W}_n is non-degenerate. We consider the following commutative diagram to obtain our results:

$$\begin{array}{ccc} \widetilde{W}_{n-k} \times S_k & \xrightarrow{\text{Ind}} & \widetilde{W}_n \\ \downarrow \text{Ind} & & \downarrow \text{Ind} \\ W_{n-k} \times S_k & \xrightarrow{\text{Ind}} & W_n \end{array}$$

- (1) Suppose $(\alpha^0, \alpha^1) \vdash (n-k)$ with $\alpha^0 \neq \alpha^1$ and $(\beta^0, \beta^1) \vdash n$ with $\beta^0 \neq \beta^1$. Additionally let $\delta \vdash k$. Then the multiplicity of $\chi^{\{\beta^0, \beta^1\}}$ in $\text{Ind}_{\widetilde{W}_{n-k} \times S_k}^{\widetilde{W}_n} (\chi^{\{\alpha^0, \alpha^1\}} \otimes \chi^\delta)$ is equal to the multiplicity of $\chi^{(\beta^0, \beta^1)}$ plus the multiplicity of $\chi^{(\beta^1, \beta^0)}$ in $\text{Ind}_{W_{n-k} \times S_k}^{W_n} (\chi^{(\alpha^0, \alpha^1)} \otimes \chi^\delta)$.
- (2) Suppose that $(\alpha, \beta) \vdash (n-k)$ with $\alpha \neq \beta$ and $(\alpha, \alpha) \vdash n$. Additionally let $\delta \vdash k$. Then the multiplicity of $\chi^{\{\alpha, \alpha\}}$ in $\text{Ind}_{\widetilde{W}_{n-k} \times S_k}^{\widetilde{W}_n} (\chi^{\{\alpha, \beta\}} \otimes \chi^\delta)$ is equal to the multiplicity of $\chi^{(\alpha, \alpha)}$ in $\text{Ind}_{W_{n-k} \times S_k}^{W_n} (\chi^{(\alpha, \beta)} \otimes \chi^\delta)$. The multiplicity of $\chi'^{\{\alpha, \alpha\}}$ is exactly the same.
- (3) Suppose that $(\alpha, \alpha) \vdash (n-k)$ and $(\alpha, \beta) \vdash n$ with $\alpha \neq \beta$. Additionally let $\delta \vdash k$. Then the multiplicity of $\chi^{\{\alpha, \beta\}}$ in $\text{Ind}_{\widetilde{W}_{n-k} \times S_k}^{\widetilde{W}_n} (\chi^{\{\alpha, \alpha\}} \otimes \chi^\delta)$ is equal to

the multiplicity of $\chi^{(\alpha,\beta)}$ in $\text{Ind}_{W_{n-k} \times S_k}^{W_n}(\chi^{(\alpha,\alpha)} \otimes \chi^\delta)$. The same statement is true with $\chi^{\{\alpha,\alpha\}}$ replaced with $\chi'^{\{\alpha,\alpha\}}$.

4. THE BRAUER HOMOMORPHISM

Let G be a finite group and M an $\mathcal{O}G$ -module. The Brauer homomorphism $Br_P^G(M)$ of M with respect to some p -subgroup P of G is the natural surjection:

$$Br_P^G : M^P \rightarrow M^P / (\sum_{Q < P} Tr_Q^P(M^Q) + \mathcal{J}M^P)$$

If M is indecomposable with trivial source then $Br_P^G(M^P) \neq 0$ if and only if M has a vertex containing P .

We consider the specific case where $M = \mathcal{O}G$ with the action of G given by conjugation. In this case $Br_P^G(M^P) \cong kC_G(P)$ as $kN_G(P)$ -modules with Br_P^G given by:

$$Br_P^G(\sum_{g \in G} \alpha_g g) = \sum_{g \in C_G(P)} \overline{\alpha_g} g$$

Now if H is a subgroup of G containing $N_G(P)$ and b and c are block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively such that the corresponding blocks both have defect group P , then $\mathcal{O}Hc$ is described as the Brauer correspondent of $\mathcal{O}Gb$ in H if $Br_P^G(b) = Br_P^H(c)$. When $H = N_G(P)$ we describe $\mathcal{O}Hc$ simply as the Brauer correspondent.

5. THE FINITE CLASSICAL GROUPS

Let q be a power of a prime. We describe each of the finite classical groups $G_m(q)$ in question in two ways. First we view $G_m(q)$ as the fixed points of some algebraic group $\mathbf{G}_m(\mathbf{q})$ under a Frobenius morphism F and secondly as the group of linear maps that preserve or scale some bilinear form over \mathbb{F}_q . For both these descriptions we follow G. Hiss and K. Kessar[12, Section 4].

5.1. Forms. Let W be an m -dimensional vector space over \mathbb{F}_q . We describe the following bilinear forms on W . We require q to be odd for the symplectic and orthogonal forms only.

Unitary: To define a unitary form we require q to be a square, say $q = q_0^2$. We want a non-degenerate sesquilinear form on W . Up to isomorphism there is one such form:

$$\langle u, v \rangle = u_1 v_m^{q_0} + \cdots + u_m v_1^{q_0}$$

Symplectic: We want a non-degenerate anti-symmetric bilinear form on W . In this case m must be even. Say $m = 2n$. Up to isomorphism there is one such form:

$$\langle u, v \rangle = (u_1 v_{2n} + \cdots + u_n v_{n+1}) - (u_{n+1} v_n + \cdots + u_{2n} v_1)$$

Orthogonal: We want a non-degenerate symmetric bilinear form on W . There are two non-isomorphic such forms:

$$\langle u, v \rangle = u_1 v_m + u_2 v_{m-1} + \cdots + u_{m-1} v_2 + u_m v_1$$

and

$$\langle u, v \rangle = u_1 v_1 + \delta u_m v_m + (u_2 v_{m-1} + u_3 v_{m-2} + \cdots + u_{m-2} v_3 + u_{m-1} v_2)$$

where $-\delta$ is a non-square in \mathbb{F}_q .

These two forms are known as type 1 and type -1 respectively.

Throughout this section if W is a vector space over K with some bilinear form g then we introduce the sets [10, Section 1]:

$$\begin{aligned} I(W, g) &= \{x \in GL(W) \mid g(xu, xv) = g(u, v), \forall u, v \in W\} \\ I_0(W, g) &= I(W, g) \cap SL(W) \end{aligned}$$

When $\dim(W)$ is even, say $2n$, then we define:

$$\begin{aligned} J(W, g) &= \{x \in GL(W) \mid \exists \lambda_x \in K, g(xu, xv) = \lambda_x g(u, v), \forall u, v \in W\} \\ J_0(W, g) &= \{x \in J(W, g) \mid \det(x) = \lambda_x^n\} \end{aligned}$$

5.2. Description of Groups. First we fix an algebraic closure $\overline{\mathbb{F}_q}$ of \mathbb{F}_q and introduce the automorphism:

$$\begin{aligned} GL_n(\overline{\mathbb{F}_q}) &\rightarrow GL_n(\overline{\mathbb{F}_q}) \\ M &\mapsto M^{[q]} \end{aligned}$$

Where $M^{[q]}$ is obtained from the matrix M by raising all its entries to the power q . In all cases except where stated F will be the above automorphism.

When q is odd we fix a non-square $\delta \in \mathbb{F}_q$. We will use J_n to denote the $n \times n$ matrix with 1s along the antidiagonal and 0s elsewhere, J'_{2n} the matrix $\begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$ and J''_{2n}

the matrix $\begin{pmatrix} 0 & 0 & 0 & J_{n-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta & 0 \\ J_{n-1} & 0 & 0 & 0 \end{pmatrix}$.

Let \mathbf{V} be an m -dimensional vector space over $\overline{\mathbb{F}_q}$ with basis $\{e_1, \dots, e_m\}$ and V the \mathbb{F}_q -vector space given by the \mathbb{F}_q -span of $\{e_1, \dots, e_m\}$. f will be a bilinear form on V but we can also consider it as a bilinear form \mathbf{f} on \mathbf{V} in a natural way.

We allow q to be even in our description of $GL_n(q)$ and $U_n(q)$ but assume q is odd for all the other groups. This convention will hold throughout this paper.

$GL_n(\mathbf{q})$: Let $\mathbf{G} = GL_n(\overline{\mathbb{F}_q})$. Then $G = GL_n(q) = \mathbf{G}^F$.
 $GL_n(q)$ has order $q^{\frac{n(n-1)}{2}} \prod_{i=1}^n (q^i - 1)$.

$U_n(\mathbf{q})$: Here we require q to be a square, say $q = q_0^2$. We set $\mathbf{G} = GL_n(\overline{\mathbb{F}_q})$ and set F to be the automorphism:

$$F : M \mapsto J_n M^{-t[q_0]} J_n$$

Then $G = U_n(q) = \mathbf{G}^F$.

Alternatively if $m = n$ and

$$f(u, v) = u_1 v_n^{q_0} + \cdots + u_n v_1^{q_0}$$

then $U_n(q) = I(V, f) = I(\mathbf{V}, \mathbf{f})$.

$U_n(q)$ has order $q_0^{\frac{n(n-1)}{2}} \prod_{i=1}^n (q_0^i - (-1)^i)$ [20, Section 2.6].

$\mathrm{Sp}_{2n}(\mathbf{q})$, $\mathrm{CSp}_{2n}(\mathbf{q})$:

$$\mathrm{Sp}_{2n}(\overline{\mathbb{F}_q}) = \{x \in \mathrm{GL}_{2n}(\overline{\mathbb{F}_q}) \mid x^t J'_{2n} x = J'_{2n}\}$$

$$\mathrm{CSp}_{2n}(\overline{\mathbb{F}_q}) = \{x \in \mathrm{GL}_{2n}(\overline{\mathbb{F}_q}) \mid x^t J'_{2n} x = \lambda_x J'_{2n}, \lambda_x \in \overline{\mathbb{F}_q}\}$$

For $\mathbf{G} = \mathrm{Sp}_{2n}(\overline{\mathbb{F}_q})$, $\mathrm{CSp}_{2n}(\overline{\mathbb{F}_q})$ we have $G = \mathbf{G}^F = \mathrm{Sp}_{2n}(q)$, $\mathrm{CSp}_{2n}(q)$ respectively.

Alternatively if $m = 2n$ and

$$f(u, v) = (u_1 v_{2n} + \cdots + u_n v_{n+1}) - (u_{n+1} v_n + \cdots + u_{2n} v_1)$$

then

$$\mathrm{Sp}_{2n}(q) = I(V, f)$$

$$\mathrm{Sp}_{2n}(\overline{\mathbb{F}_q}) = I(\mathbf{V}, \mathbf{f})$$

$$\mathrm{CSp}_{2n}(q) = J(V, f)$$

$$\mathrm{CSp}_{2n}(\overline{\mathbb{F}_q}) = J(\mathbf{V}, \mathbf{f})$$

$\mathrm{CSp}_{2n}(q)$ has order $q^{n^2}(q-1) \prod_{i=1}^n (q^{2i} - 1)$ [20, Section 2.6].

$\mathrm{O}_{2n+1}(\mathbf{q})$, $\mathrm{SO}_{2n+1}(\mathbf{q})$, $\mathrm{CO}_{2n+1}(\mathbf{q})$, $\mathrm{CSO}_{2n+1}(\mathbf{q})$:

$$\mathrm{O}_{2n+1}(\overline{\mathbb{F}_q}) = \{x \in \mathrm{GL}_{2n+1}(\overline{\mathbb{F}_q}) \mid x^t J_{2n+1} x = J_{2n+1}\}$$

$$\mathrm{SO}_{2n+1}(\overline{\mathbb{F}_q}) = \mathrm{O}_{2n+1}(\overline{\mathbb{F}_q}) \cap \mathrm{SL}_{2n+1}(\overline{\mathbb{F}_q})$$

For $\mathbf{G} = \mathrm{O}_{2n+1}(\overline{\mathbb{F}_q})$, $\mathrm{SO}_{2n+1}(\overline{\mathbb{F}_q})$ we have $G = \mathbf{G}^F = \mathrm{O}_{2n+1}(q)$, $\mathrm{SO}_{2n+1}(q)$ respectively.

Alternatively if $m = 2n + 1$ and

$$f(u, v) = u_1 v_{2n+1} + \cdots + u_{2n+1} v_1$$

then

$$\begin{aligned} O_{2n+1}(q) &= I(V, f) & O_{2n+1}(\overline{\mathbb{F}}_q) &= I(\mathbf{V}, \mathbf{f}) \\ SO_{2n+1}(q) &= I_0(V, f) & SO_{2n+1}(\overline{\mathbb{F}}_q) &= I_0(\mathbf{V}, \mathbf{f}) \end{aligned}$$

$SO_{2n+1}(q)$ has order $q^{n^2} \prod_{i=1}^n (q^{2i} - 1)$ [20, Section 2.6].

We note that when $\dim(V)$ is odd although there are two non-isomorphic non-degenerate orthogonal forms on V they produce isomorphic groups.

$\mathbf{O}_{2n}^+(\mathbf{q})$, $\mathbf{SO}_{2n}^+(\mathbf{q})$, $\mathbf{CO}_{2n}^+(\mathbf{q})$, $\mathbf{CSO}_{2n}^+(\mathbf{q})$:

$$\begin{aligned} O_{2n}^+(\overline{\mathbb{F}}_q) &= \{x \in GL_{2n}(\overline{\mathbb{F}}_q) | x^t J_{2n} x = J_{2n}\} \\ SO_{2n}^+(\overline{\mathbb{F}}_q) &= O_{2n}^+(\overline{\mathbb{F}}_q) \cap SL_{2n}(\overline{\mathbb{F}}_q) \end{aligned}$$

For $\mathbf{G} = O_{2n}^+(\overline{\mathbb{F}}_q)$, $SO_{2n}^+(\overline{\mathbb{F}}_q)$ we have $G = \mathbf{G}^F = O_{2n}^+(q)$, $SO_{2n}^+(q)$ respectively.

$$\begin{aligned} CO_{2n}^+(\overline{\mathbb{F}}_q) &= \{x \in GL_{2n}(\overline{\mathbb{F}}_q) | x^t J_{2n} x = \lambda_x J_{2n}, \lambda_x \in \overline{\mathbb{F}}_q\} \\ CSO_{2n}^+(\overline{\mathbb{F}}_q) &= \{x \in CO_{2n}^+(\overline{\mathbb{F}}_q) | \lambda_x^n = \det(x)\} \end{aligned}$$

For $\mathbf{G} = CO_{2n}^+(\overline{\mathbb{F}}_q)$, $CSO_{2n}^+(\overline{\mathbb{F}}_q)$ we have $G = \mathbf{G}^F = CO_{2n}^+(q)$, $CSO_{2n}^+(q)$ respectively.

Alternatively if $m = 2n$ and

$$f(u, v) = u_1 v_{2n} + \cdots + u_{2n} v_1$$

then

$$\begin{aligned} O_{2n}^+(q) &= I(V, f) & O_{2n}^+(\overline{\mathbb{F}}_q) &= I(\mathbf{V}, \mathbf{f}) \\ SO_{2n}^+(q) &= I_0(V, f) & SO_{2n}^+(\overline{\mathbb{F}}_q) &= I_0(\mathbf{V}, \mathbf{f}) \\ CO_{2n}^+(q) &= J(V, f) & CO_{2n}^+(\overline{\mathbb{F}}_q) &= J(\mathbf{V}, \mathbf{f}) \\ CSO_{2n}^+(q) &= J_0(V, f) & CSO_{2n}^+(\overline{\mathbb{F}}_q) &= J_0(\mathbf{V}, \mathbf{f}) \end{aligned}$$

$SO_{2n}^+(q)$ has order $q^{n(n-1)}(q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1)$ [20, Section 2.6].
 $CSO_{2n}^+(q)$ has order $q^{n(n-1)}(q^n - 1)(q - 1) \prod_{i=1}^{n-1} (q^{2i} - 1)$ [20, Section 2.6].

$\mathbf{O}_{2n}^-(\mathbf{q})$, $\mathbf{SO}_{2n}^-(\mathbf{q})$, $\mathbf{CO}_{2n}^-(\mathbf{q})$, $\mathbf{CSO}_{2n}^-(\mathbf{q})$:

$$\begin{aligned} O_{2n}^-(\overline{\mathbb{F}}_q) &= \{x \in GL_{2n}(\overline{\mathbb{F}}_q) | x^t J_{2n}'' x = J_{2n}''\} \\ SO_{2n}^-(\overline{\mathbb{F}}_q) &= O_{2n}^-(\overline{\mathbb{F}}_q) \cap SL_{2n}(\overline{\mathbb{F}}_q) \end{aligned}$$

For $\mathbf{G} = O_{2n}^-(\overline{\mathbb{F}}_q)$, $SO_{2n}^-(\overline{\mathbb{F}}_q)$ we have $G = \mathbf{G}^F = O_{2n}^-(q)$, $SO_{2n}^-(q)$ respectively.

$$CO_{2n}^-(\overline{\mathbb{F}_q}) = \{x \in GL_{2n}(\overline{\mathbb{F}_q}) | x^t J_{2n}'' x = \lambda_x J_{2n}'', \lambda_x \in \overline{\mathbb{F}_q}\}$$

$$CSO_{2n}^-(\overline{\mathbb{F}_q}) = \{x \in CO_{2n}^-(\overline{\mathbb{F}_q}) | \lambda_x^n = \det(x)\}$$

For $\mathbf{G} = CO_{2n}^-(\overline{\mathbb{F}_q})$, $CSO_{2n}^-(\overline{\mathbb{F}_q})$ we have $G = \mathbf{G}^F = CO_{2n}^-(q)$, $CSO_{2n}^-(q)$ respectively.

Alternatively if $m = 2n$ and

$$f(u, v) = u_1 v_{2n} + \cdots + u_n v_n + \delta u_{n+1} v_{n+1} + \cdots + u_{2n} v_1$$

then

$$\begin{array}{ll} O_{2n}^-(q) = I(V, f) & O_{2n}^-(\overline{\mathbb{F}_q}) = I(\mathbf{V}, \mathbf{f}) \\ SO_{2n}^-(q) = I_0(V, f) & SO_{2n}^-(\overline{\mathbb{F}_q}) = I_0(\mathbf{V}, \mathbf{f}) \\ CO_{2n}^-(q) = J(V, f) & CO_{2n}^-(\overline{\mathbb{F}_q}) = J(\mathbf{V}, \mathbf{f}) \\ CSO_{2n}^-(q) = J_0(V, f) & CSO_{2n}^-(\overline{\mathbb{F}_q}) = J_0(\mathbf{V}, \mathbf{f}) \end{array}$$

$$SO_{2n}^-(q) \text{ has order } q^{n(n-1)}(q^n + 1) \prod_{i=1}^{n-1} (q^{2i} - 1) [20, \text{Section 2.6}].$$

$$CSO_{2n}^-(q) \text{ has order } q^{n(n-1)}(q^n + 1)(q - 1) \prod_{i=1}^{n-1} (q^{2i} - 1) [20, \text{Section 2.6}].$$

5.3. Clifford Groups. For our description of Clifford groups we follow P. Fong and B. Srinivasan[10, Section 2]. Let V be a vector space over K of finite dimension greater than 1 endowed with a non-degenerate quadratic form Q . Let $<, >$ be the corresponding symmetric bilinear form. The Clifford algebra $\mathcal{C}(V)$ is the K -algebra generated by V subject to all the linear relations in V as well as the additional condition:

$$v^2 = Q(v)1 \text{ for all } v \in V$$

$\mathcal{C}(V)$ has a \mathbb{Z}_2 -grading given by demanding all the non-zero $v \in V$ are odd. We use $\mathcal{C}_+(V)$ to denote the even part. Now define the Clifford group $Cl(V)$ and special Clifford group $Cl_+(V)$ as follows:

$$Cl(V) = \{x \in \mathcal{C}(V)^\times | xVx^{-1} = V\}$$

$$Cl_+(V) = \{x \in \mathcal{C}_+(V)^\times | xVx^{-1} = V\}$$

Now if $x \in Cl(V)$ and $v \in V$ then:

$$Q(xvx^{-1}) = (xvx^{-1})^2 = xv^2x^{-1} = xQ(v)x^{-1} = Q(v)$$

Therefore conjugation by an element in $Cl(V)$ preserves $<, >$ on V . This gives us the map:

$$Cl(V) \rightarrow O(V)$$

where $O(V)$ is the group of linear maps on V that preserve $<, >$.

When we restrict to $Cl_+(V)$ we get the short exact sequence:

$$0 \longrightarrow K^\times \longrightarrow Cl_+(V) \xrightarrow{\pi} SO(V) \longrightarrow 0$$

Let q be the power of an odd prime. We use $Cl_{2n+1}(q)$ (respectively $Cl_{2n}^+(q)$, $Cl_{2n}^-(q)$) to denote the special Clifford groups defined over \mathbb{F}_q with respect to the bilinear form given by the Gram matrix J_{2n+1} (respectively J_{2n} , J_{2n}'').

Similarly we can define $Cl_{2n+1}(\overline{\mathbb{F}_q})$, $Cl_{2n}^+(\overline{\mathbb{F}_q})$ and $Cl_{2n}^-(\overline{\mathbb{F}_q})$ over $\overline{\mathbb{F}_q}$.

If \mathbf{V} is the $\overline{\mathbb{F}_q}$ -vector space with basis $\{e_1 \dots e_m\}$ then we have an \mathbb{F}_q -vector space automorphism of \mathbf{V} given by:

$$\sum \alpha_i e_i \mapsto \sum \alpha_i^q e_i$$

This extends to an automorphism F of $Cl_{2n+1}(\overline{\mathbb{F}_q})$ (respectively $Cl_{2n}^+(\overline{\mathbb{F}_q})$, $Cl_{2n}^-(\overline{\mathbb{F}_q})$) (where $\{e_1 \dots e_m\}$ is the basis with respect to which the Gram matrices are taken). Then we have:

$$\begin{aligned} Cl_{2n+1}(q) &= Cl_{2n+1}(\overline{\mathbb{F}_q})^F \\ Cl_{2n}^+(q) &= Cl_{2n}^+(\overline{\mathbb{F}_q})^F \\ Cl_{2n}^-(q) &= Cl_{2n}^-(\overline{\mathbb{F}_q})^F \end{aligned}$$

5.4. Levi Subgroups. We will now describe certain subgroups of the finite classical and Clifford groups described above as in [12, Section 4.4].

GL_n(q): If we have positive integers (n_1, n_2, \dots, n_t) with $\sum_i n_i = n$ then $GL_{n_1}(q) \times \dots \times GL_{n_t}(q)$ embeds in $GL_n(q)$ in the natural way:

$$\begin{aligned} GL_{n_1}(q) \times \dots \times GL_{n_t}(q) &\hookrightarrow GL_n(q) \\ A_1 \times \dots \times A_t &\mapsto \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_t \end{pmatrix} \end{aligned}$$

We will use $L_{(n_1, \dots, n_t)}(q)$ to denote this Levi subgroup.

U_n(q), Sp_{2n}(q), CSp_{2n}(q), SO_{2n+1}(q), SO_{2n}[±](q), CSO_{2n}[±](q): Let $G_m(q)$ be one of the above groups and (n_1, n_2, \dots, n_t) positive integers with $2s < m$ where $s = \sum_i n_i$, then $GL_{n_1}(q) \times \dots \times GL_{n_t}(q) \times G_{m-2s}(q)$ embeds in $G_m(q)$ via the map:

$$GL_{n_1}(q) \times \cdots \times GL_{n_t}(q) \times G_{m-2s}(q) \hookrightarrow G_m(q)$$

$$A_1 \times \cdots \times A_t \times B \mapsto \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_t & & & & \\ & & & B & & & \\ & & & & A'_t & & \\ & & & & & \ddots & \\ & & & & & & A'_1 \end{pmatrix}$$

where the A'_i are chosen to stay in $G_m(q)$. So $A'_i = J_{n_i} A_i^{-t[q_0]} J_{n_i}$ for the unitary group and $A'_i = \lambda_B J_{n_i} A_i^{-t[q]} J_{n_i}$ in all other cases. Note that λ_B is only needed for the conformal groups and we set $\lambda_B = 1$ for all the other groups. We will use $L_{m,(n_1,\dots,n_t)}(q)$ to denote this Levi subgroup of $G_m(q)$.

$\mathbf{Cl}_{2n+1}(\mathbf{q})$, $\mathbf{Cl}_{2n}^\pm(\mathbf{q})$: Let $Cl_m^{(\pm)}(q)$ denote one of the above groups and $SO_m^{(\pm)}(q)$ the corresponding special orthogonal group giving us the exact sequence.

$$0 \longrightarrow \mathbb{F}_q \longrightarrow Cl_m^{(\pm)}(q) \longrightarrow SO_m^{(\pm)}(q) \longrightarrow 0$$

Then if we have positive integers (n_1, n_2, \dots, n_t) with $2s < m$ where $\sum_i n_i = s$ then we have $L_{m,(n_1,\dots,n_t)}(q) \cong GL_{n_1}(q) \times \cdots \times GL_{n_t}(q) \times SO_{m-2s}^{(\pm)}(q)$ is a Levi subgroup of $SO_m^{(\pm)}(q)$ as described above. The pre-image of $L_{m,(n_1,\dots,n_t)}(q)$ in $Cl_m^{(\pm)}(q)$ is isomorphic to $GL_{n_1}(q) \times \cdots \times GL_{n_t}(q) \times Cl_{m-2s}^{(\pm)}(q)$ and we also denote it by $L_{m,(n_1,\dots,n_t)}(q)$.

6. REPRESENTATION THEORY OF THE FINITE CLASSICAL GROUPS

6.1. Semisimple elements. Semisimple elements will be important when we describe the ordinary characters of the finite classical groups. By a semisimple element we mean a matrix whose minimal polynomial has no repeated roots. Equivalently they are the matrices that are diagonalisable over some field extension. If Γ is a polynomial let d_Γ denote the degree of Γ . Finally we let Δ be the set of monic irreducible polynomials over \mathbb{F}_q with non-zero roots. We follow P. Fong and B. Srinivasen in [9, Section 1] and [10, Section 1] for our description of semisimple elements.

$\mathbf{GL}_n(\mathbf{q})$: A conjugacy class of semisimple elements in $GL_n(q)$ is completely defined by their common characteristic polynomial. If $\Gamma \in \Delta$ let $m_\Gamma(s)$ be the multiplicity of Γ in the characteristic polynomial of s . So we have the function

$$\begin{aligned} \Delta &\rightarrow \mathbb{N}_0 \\ \Gamma &\mapsto m_\Gamma(s) \end{aligned}$$

with $\sum_{\Gamma \in \Delta} m_{\Gamma}(s) \cdot d_{\Gamma} = n$. Conversely any such function uniquely defines a semisimple conjugacy class in $GL_n(q)$.

$U_n(\mathbf{q})$: A conjugacy class of semisimple elements in $U_n(q)$ is also completely defined by their common characteristic polynomial. However, if $s \in U_n(q)$ and ω is a root of the characteristic polynomial of s then ω^{-q_0} is also a root occurring with the same multiplicity. With that in mind we define the following involution on Δ :

$$\begin{aligned}\Gamma &= (X^m + a_{m-1}X^{m-1} + \cdots + a_1X + a_0) \mapsto \\ \bar{\Gamma} &= a_0^{-q_0}(a_0^{q_0}X^m + a_1^{q_0}X^{m-1} + \cdots + a_{m-1}^{q_0}X + 1)\end{aligned}$$

So $\bar{\Gamma}$ is the unique monic polynomial with roots those of Γ raised to the power $-q_0$.

$$\begin{aligned}\Lambda_1 &= \{\Gamma \mid \Gamma \in \Delta, \bar{\Gamma} = \Gamma\} \\ \Lambda_2 &= \{\Gamma \mid \Gamma \in \Delta, \bar{\Gamma} \neq \Gamma\} \\ \Lambda &= \Lambda_1 \cup \Lambda_2\end{aligned}$$

If $\Gamma \in \Lambda$ let $m_{\Gamma}(s)$ be the multiplicity of Γ in the characteristic polynomial of s . So we have the function

$$\begin{aligned}\Lambda &\rightarrow \mathbb{N}_0 \\ \Gamma &\mapsto m_{\Gamma}(s)\end{aligned}$$

with $\sum_{\Gamma \in \Lambda} m_{\Gamma}(s) \cdot d_{\Gamma} = n$. Conversely any such function uniquely defines a semisimple conjugacy class in $U_n(q)$.

$Sp_{2n}(\mathbf{q})$: A conjugacy class of semisimple elements in $Sp_{2n}(q)$ is also completely defined by their common characteristic polynomial. However, if $s \in Sp_{2n}(q)$ and ω is a root of the characteristic polynomial of s then ω^{-1} is also a root occurring with the same multiplicity. With that in mind we define the following involution on Δ :

$$\begin{aligned}\Gamma &= (X^m + a_{m-1}X^{m-1} + \cdots + a_1X + a_0) \mapsto \\ \tilde{\Gamma} &= a_0^{-1}(a_0X^m + a_1X^{m-1} + \cdots + a_{m-1}X + 1)\end{aligned}$$

So $\tilde{\Gamma}$ is the unique monic polynomial with roots the inverses of those of Γ . Next we define the sets:

$$\begin{aligned}
\Phi_0 &= \{X - 1, X + 1\} \\
\Phi_1 &= \{\Gamma | \Gamma \in \Delta \setminus \Phi_0, \tilde{\Gamma} = \Gamma\} \\
\Phi_2 &= \{\Gamma | \tilde{\Gamma} \in \Delta, \tilde{\Gamma} \neq \Gamma\} \\
\Phi &= \Phi_0 \cup \Phi_1 \cup \Phi_2
\end{aligned}$$

If $\Gamma \in \Phi$ let $m_\Gamma(s)$ be the multiplicity of Γ in the characteristic polynomial of s . So we have the function

$$\begin{aligned}
\Phi &\rightarrow \mathbb{N}_0 \\
\Gamma &\mapsto m_\Gamma(s)
\end{aligned}$$

with $m_{X+1}(s)$ even and $\sum_{\Gamma \in \Phi} m_\Gamma(s) \cdot d_\Gamma = 2n$. Conversely any such function uniquely defines a semisimple conjugacy class in $Sp_{2n}(q)$.

SO_m^(±)(q): Let $SO_m^{(\pm)}(q)$ be any one of our special orthogonal groups. Once again if $s \in SO_m^{(\pm)}(q)$ and ω is a root of the characteristic polynomial of s then ω^{-1} is also a root occurring with the same multiplicity. So again we have the function:

$$\begin{aligned}
\Phi &\rightarrow \mathbb{N}_0 \\
\Gamma &\mapsto m_\Gamma(s)
\end{aligned}$$

with $\sum_{\Gamma \in \Phi} m_\Gamma(s) \cdot d_\Gamma = m$ and $m_{X+1}(s)$ even. This does not uniquely determine a conjugacy class however. If both $m_{X-1}(s)$ and $m_{X+1}(s)$ are non-zero this function defines 2 conjugacy classes, otherwise we just get 1 conjugacy class. Later on the semisimple elements we consider will all have $m_{X+1}(s) = 0$.

Cl_m^(±)(q): [10, Section 2] We have the surjection described in 5.3:

$$\pi : Cl_m^{(\pm)}(q) \twoheadrightarrow SO_m^{(\pm)}(q)$$

An element of $Cl_m^{(\pm)}(q)$ is described as semisimple if its image in $SO_m^{(\pm)}(q)$ is semisimple. Let $\pi(t) = s \in SO_m^{(\pm)}(q)$ be semisimple and C its conjugacy class in $SO_m^{(\pm)}(q)$. If $m_{X-1}(s)$ and $m_{X+1}(s)$ are both non-zero then $\pi^{-1}(C)$ is the union of $\frac{q-1}{2}$ conjugacy classes in $Cl_m^{(\pm)}(q)$ and t is conjugate to $-t$. Otherwise $\pi^{-1}(C)$ is the union of $q-1$ conjugacy classes in $Cl_m^{(\pm)}(q)$ and no 2 distinct pre-images of s are conjugate.

From now on we will use $m_\Gamma(t)$ to denote $m_\Gamma(s)$.

For any semisimple element s we have a corresponding decomposition of $V = \mathbb{F}_q^m$. For each $\Gamma \in \Phi$ (or Δ for $GL_n(q)$ or Λ for $U_n(q)$) we define V_Γ to be the null space of $\Gamma(s)$ (or $\Gamma(\pi(s))$ in the case of the special Clifford groups).

$$V = \oplus_{\Gamma} V_\Gamma$$

is then an orthogonal decomposition of V .

6.2. Characters of Finite Classical groups. We will describe labels for the ordinary irreducible characters of some of the finite classical groups. We follow P. Fong and B. Srinivasan in [9, Section 1] and [10, Section 2] for the description of the characters. We leave out $Sp_{2n}(q)$ and $SO^\pm(q)$, the appropriate characters for these groups will be described in section 8.4. First of all we will describe a subset of all the ordinary characters called the unipotent characters.

The unipotent characters of $GL_n(q)$ and $U_n(q)$ are both labeled by partitions of n (for example (n) corresponds to the trivial character of both groups).

The unipotent characters of $CSp_{2n}(q)$ and $SO_{2n+1}(q)$ are both labeled by symbols with rank n and odd defect.

The unipotent characters of $CSO_{2n}^+(q)$ are labeled by symbols with rank n and defect $\equiv 0 \pmod{4}$ with the added rule that degenerate symbols label 2 characters.

The unipotent characters of $CSO_{2n}^-(q)$ are labeled by symbols with rank n and defect $\equiv 2 \pmod{4}$.

Consider the following table:

G	G^*
$GL_n(q)$	$GL_n(q)$
$U_n(q)$	$U_n(q)$
$SO_{2n+1}(q)$	$Sp_{2n}(q)$
$CSp_{2n}(q)$	$Cl_{2n+1}(q)$
$CSO_{2n}^+(q)$	$Cl_{2n}^+(q)$
$CSO_{2n}^-(q)$	$Cl_{2n}^-(q)$

Let $s \in G^*$ be semisimple with corresponding decomposition $V = \oplus_{\Gamma} V_{\Gamma}$. We define $\Psi_{\Gamma}(s)$ as follows:

If G^* is a general linear or unitary group then $\Psi_{\Gamma}(s) = \{\text{partitions of } m_{\Gamma}(s)\}$.

Now we assume that G^* is not a general linear or unitary group.

For $\Gamma \in \Phi_0$ let $\Psi_{\Gamma}(s) = \{\text{symbols of rank } \lfloor \frac{m_{\Gamma}(s)}{2} \rfloor\}$ subject to the following conditions:

- (1) If the form induced on V_{Γ} is symplectic or orthogonal of odd dimension, then the symbols have odd defect.
- (2) If the form induced on V_{Γ} is orthogonal of even dimension and type 1, then the symbols have defect $\equiv 0 \pmod{4}$. Moreover, degenerate symbols are counted twice. If λ is such a degenerate symbol then we say $\lambda, \lambda' \in \Psi_{\Gamma}(s)$.
- (3) If the form induced on V_{Γ} is orthogonal of even dimension and type -1, then the symbols have $\equiv 2 \pmod{4}$.

For $\Gamma \in \Phi_1 \cup \Phi_2$ let $\Psi_{\Gamma}(s) = \{\text{partitions of } m_{\Gamma}(s)\}$.

Now we set $\Psi(s) = \prod_{\Gamma} \Psi_{\Gamma}(s)$.

The characters of G are labeled by a semisimple element s of G^* together with a $\lambda \in \Psi(s)$.

We denote this character $\chi_{s,\lambda}$. $\chi_{s,\lambda}$ and $\chi_{t,\mu}$ represent the same character if and only if s is conjugate to t in G^* and $\lambda = \mu$.

Now if s does not have -1 as an eigenvalue then we can label $\chi_{s,\lambda}$ by a unipotent character of $C_{G^*}(s)^*$ [9, Section 1], [10, Section 4].

We will now describe $C_{G^*}(s)$ for all relevant G to make this labeling clear.

GL_n(q): Let $G^* = GL_n(q)$ and let $s \in G^*$ be semisimple. Then:

$$C_{G^*}(s) \cong \prod_{\Gamma \in \Delta} GL_{m_\Gamma(s)}(q^{d_\Gamma})$$

U_n(q): Let $G^* = U_n(q)$ and let $s \in G^*$ be semisimple. Then:

$$C_{G^*}(s) \cong \left(\prod_{\Gamma \in \Phi_1} U_{m_\Gamma(s)}(q^{d_\Gamma}) \right) \times \left(\prod_{\Gamma \in \Phi_2} GL_{m_\Gamma(s)}(q^{\frac{d_\Gamma}{2}}) \right)$$

Sp_{2n}(q), Cl_{2n+1}(q), Cl_{2n}[±](q): Let $G^* = G_m(q)$ be one of the above groups. Let $s \in G^*$ be semisimple.

$$C_{G^*}(s) \cong G_{m_{X-1}(s)}(q) \times \left(\prod_{\Gamma \in \Phi_1} U_{m_\Gamma(s)}(q^{d_\Gamma}) \right) \times \left(\prod_{\Gamma \in \Phi_2} GL_{m_\Gamma(s)}(q^{d_\Gamma}) \right)$$

7. HARISH-CHANDRA INDUCTION

7.1. Harish-Chandra Induction. Let p be an odd prime not dividing q and (K, \mathcal{O}, k) a p -modular system as introduced in the introduction 1. Let $G = G_m(q)$ be $GL_n(q)$, $U_n(q)$, $Sp_{2n}(q)$, $CSp_{2n}(q)$, $SO_{2n+1}(q)$, $SO_{2n}^+(q)$, $SO_{2n}^-(q)$, $CSO_{2n}^+(q)$ or $CSO_{2n}^-(q)$ and $L = L_{m,(n_1,\dots,n_t)}(q)$ (or $L = L_{(n_1,\dots,n_t)}(q)$ in the case of the general linear group) as described in section 5.4. We describe a functor $HCInd_L^G$ from $\mathcal{O}L\text{-mod}$ to $\mathcal{O}G\text{-mod}$ [8, Example 4.6(iii)].

We set U to be the subgroup of G consisting of matrices of the form:

$$\begin{pmatrix} I_{n_1} & \cdots & * \\ & \ddots & \vdots \\ & & I_{n_t} \end{pmatrix}$$

for the general linear group and

$$\begin{pmatrix} I_{n_1} & \cdots & * & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & I_{n_t} & * & * & \cdots & * \\ & & & I_{m-2s} & * & \cdots & * \\ & & & & I_{n_t} & \cdots & * \\ & & & & & \ddots & \vdots \\ & & & & & & I_{n_1} \end{pmatrix}$$

otherwise.

Now in all cases U has p' order. In other words $|U|$ is invertible in \mathcal{O} . Let U^+ be the idempotent $\frac{1}{|U|} \sum_{u \in U} u$ in $\mathcal{O}G$. L normalises U and so commutes with U^+ and so $\mathcal{O}GU^+$ is an $(\mathcal{O}G, \mathcal{O}L)$ -bimodule. Now set $HCInd_L^G$ to be the functor $\mathcal{O}GU^+ \otimes_{\mathcal{O}L} -$.

If b is a central idempotent of $\mathcal{O}G$ and c a central idempotent of $\mathcal{O}L$. We will want the functor $\mathcal{O}GbU^+ \otimes_{\mathcal{O}Lc} -$ from $\mathcal{O}Lc\text{-mod}$ to $\mathcal{O}Gb\text{-mod}$. We denote this functor $HCInd_{L,c}^{G,b}$.

7.2. Characters under Harish-Chandra Induction. Harish-Chandra induction is a functor from $\mathcal{O}L\text{-mod}$ to $\mathcal{O}G\text{-mod}$. However, we want to know what happens at the level of characters viewing Harish-Chandra induction as a functor from $KL\text{-mod}$ to $KG\text{-mod}$. Let R_L^G denote Harish-Chandra induction on characters. Again if b is a central idempotent of $\mathcal{O}G$ and c a central idempotent of $\mathcal{O}L$ then we denote by:

$$R_{L,c}^{G,b} : Irr(\mathcal{O}Lc) \rightarrow Irr(\mathcal{O}Gb)$$

the corresponding function on characters.

In this section we only consider $L = L_{m,(k)}(q)$ (or $L = L_{(k,n-k)}(q)$ for the general linear group). The reason for this is that Harish-Chandra induction is transitive. This means that if L has more general linear factors then the functor R_L^G can be calculated iteratively.

7.2.1. Unipotent Characters. First we will first look at the effect of Harish-Chandra induction on unipotent characters for the groups $GL_n(q)$, $U_n(q)$, $CSp_{2n}(q)$, $SO_{2n+1}(q)$, $CSO_{2n}^\pm(q)$ [12, Section 5.3].

$GL_n(q)$: If $G = GL_n(q)$ and $L = L_{(k,n-k)} \cong GL_k(q) \times GL_{n-k}(q)$ then the multiplicity of $\chi_{1,\gamma}$ in $R_L^G(\chi_{1,\alpha} \otimes \chi_{1,\beta})$ is $g_{\alpha,\beta}^\gamma$ (see 2.4).

$U_n(q)$, $CSp_{2n}(q)$, $SO_{2n+1}(q)$, $CSO_{2n}^-(q)$: Let $G = G_m(q)$ be one of the above groups and $L = L_{m,(k)}(q) \cong GL_k(q) \times G_{m-2k}(q)$. We use the alternative description of partitions and symbols described in section 2.3.

In all four groups the multiplicity of $\chi_{1,(s,\mu,\nu)}$ in $R_L^G(\chi_{1,\gamma} \otimes \chi_{1,(t,\alpha,\beta)})$ is 0 unless $s = t$ and in this case it is equal to the multiplicity of $\chi^{(\mu,\nu)}$ in $Ind_{S_k \times W_{v-k}}^{W_v}(\chi^\gamma \otimes \chi^{(\alpha,\beta)})$ for an appropriate v (see 3).

$CSO_{2n}^+(q)$: For $CSO_{2n}^+(q)$ we again use the description in 2.3. We then do exactly the same calculation as above unless $s = 0$. In this case the calculation is then carried out in a Weyl group of type D_m instead of B_m . In other words the multiplicity of $\chi_{1,(0,\mu,\nu)}$ in $R_L^G(\chi_{1,\gamma} \otimes \chi_{1,(0,\alpha,\beta)})$ is equal to the multiplicity of

$\chi^{\{\mu, \nu\}}$ in $\text{Ind}_{S_k \times \widetilde{W}_{v-k}}^{\widetilde{W}_v}(\chi^\gamma \otimes \chi^{\{\alpha, \beta\}})$ for an appropriate v . Note that $\chi_{1, (0, \mu, \mu)}$, $\chi'_{1, (0, \mu, \mu)}$ correspond to $\chi^{\{\mu, \mu\}}$, $\chi'^{\{\mu, \mu\}}$ respectively.

7.2.2. General Case. Now we drop the assumption that the characters are unipotent. However, we keep the assumption that our semisimple labels don't have -1 as an eigenvalue. Let $s \in L^*$ be semisimple. We can then use 6.2 to calculate $R_L^G(\chi_{s, \lambda})$ by passing to Harish-Chandra induction for unipotent characters from $C_{L^*}(s)^*$ to $C_{G^*}(s)^*$ [12, Section 5.4].

GL_n(q): Let $G = GL_n(q)$ and $L = L_{(k, n-k)} \cong GL_k(q) \times GL_{n-k}(q)$. If $\chi_{s_1, \lambda_1} \otimes \chi_{s_2, \lambda_2}$ is a character of L then

$$C_{L^*}(s_1 \times s_2) \cong \prod_{\Gamma \in \Delta} (GL_{m_\Gamma(s_1)}(q^{d_\Gamma}) \times GL_{m_\Gamma(s_2)}(q^{d_\Gamma}))$$

So for each $\Gamma \in \Delta$ we do Harish-Chandra induction to get a sum of unipotent character of

$$GL_{m_\Gamma(s_1) + m_\Gamma(s_2)}(q^{d_\Gamma})$$

We now have a sum of unipotent characters for $C_{G^*}(s_1 \times s_2)^*$ and hence a character of G . This character is $R_L^G(\chi_{s_1, \lambda_1} \otimes \chi_{s_2, \lambda_2})$.

U_n(q): Let $G = U_n(q)$ and $L = L_{n, (k)} \cong GL_k(q) \times U_{n-2k}(q)$. If $\chi_{s_1, \lambda_1} \otimes \chi_{s_2, \lambda_2}$ is a character of L then

$$C_{L^*}(s_1 \times s_2) \cong \prod_{\Gamma \in \Lambda_1} (GL_{m_\Gamma(s_1)}(q^{d_\Gamma}) \times U_{m_\Gamma(s_2)}(q^{d_\Gamma})) \times \prod_{\Gamma \in \Lambda_2} (GL_{m_\Gamma(s_1)}(q^{d_\Gamma}) \times GL_{m_{\Gamma}(s_1)}(q^{d_\Gamma}) \times GL_{m_{\Gamma}(s_2)}(q^{d_\Gamma}))$$

Then for each $\Gamma \in \Lambda_1$ we do Harish-Chandra induction to get a sum of unipotent characters of

$$U_{2m_\Gamma(s_1) + m_\Gamma(s_2)}(q^{d_\Gamma})$$

and for each $\Gamma \in \Lambda_2$ we do Harish-Chandra induction twice to get a sum of unipotent characters of

$$GL_{m_\Gamma(s_1) + m_{\Gamma}(s_1) + m_{\Gamma}(s_2)}(q^{d_\Gamma})$$

We now have a sum of unipotent characters of $C_{G^*}(s_1 \times s_2)^*$ and hence a character of G . This character is $R_L^G(\chi_{s_1, \lambda_1} \otimes \chi_{s_2, \lambda_2})$.

SO_{2n+1}(q), CSp_{2n}(q), CSO_{2n}⁺(q), CSO_{2n}⁻(q): Let $G = G_m(q)$ be $SO_{2n+1}(q)$, $CSp_{2n}(q)$, $CSO_{2n}^+(q)$ or $CSO_{2n}^-(q)$ and $L = L_{m, (k)} \cong GL_k(q) \times G_{m-2k}(q)$. If $\chi_{s_1, \lambda_1} \otimes \chi_{s_2, \lambda_2}$ is a character of L then

$$C_{L^*}(s_1 \times s_2) \cong (GL_{m_{X-1}(s_1)}(q) \times G_{m_{X-1}(s_2)}^*(q)) \times \prod_{\Gamma \in \Psi_1} (GL_{m_\Gamma(s_1)}(q^{d_\Gamma}) \times U_{m_\Gamma(s_1)}(q^{d_\Gamma})) \times \prod_{\Gamma \in \Psi_2} (GL_{m_\Gamma(s_1)}(q^{d_\Gamma}) \times GL_{m_{\Gamma}(s_1)}(q^{d_\Gamma}) \times GL_{m_{\Gamma}(s_2)}(q^{d_\Gamma}))$$

Once again we do Harish-Chandra induction on unipotent characters to obtain a sum of unipotent characters of $C_{G^*}(s_1 \times s_2)^*$ and hence a character of G . This character is $R_L^G(\chi_{s_1, \lambda_1} \otimes \chi_{s_2, \lambda_2})$.

7.3. Characters with Equal Dimension. We can use Harish-Chandra induction to show that some pairs of characters have the same dimension. This will be useful later when we will be performing some calculations.

Lemma 7.3.1. *The following pairs of characters have the same dimensions.*

- (1) $\chi_{s, \lambda}$ and $\chi_{s^{-q_0}, \lambda^{-q_0}}$ of $GL_k(q)$ where $q = q_0^2$ and $\lambda_{\Gamma}^{-q_0} = \lambda_{\bar{\Gamma}}$ (see 6.1).
- (2) $\chi_{s, \lambda}$ and $\chi_{s^{-1}, \lambda^{-1}}$ of $GL_k(q)$ where $\lambda_{\Gamma}^{-1} = \lambda_{\bar{\Gamma}}$ (see section 6.1).
- (3) $\chi_{s, \lambda}$ and $\chi_{s, \lambda'}$ of $CSO_{2n}^+(q)$ where $s \in \mathbb{F}_q$ and λ is a degenerate symbol.

Proof. We prove all three results by considering Harish-Chandra induction of a pair of characters from L to G that give the same character and noting that the dimension is always multiplied by $[G : LU]$ (see 7).

- (1) We set $G = U_{n+2k}(q)$ and $L = GL_k(q) \times U_n(q)$. We Harish-Chandra induce $\chi_{s, \lambda} \otimes \chi$ and $\chi_{s^{-q_0}, \lambda^{-q_0}} \otimes \chi$ where χ is any unipotent character of $U(q)$.
- (2) We set $G = CSp_{2(n+k)}(q)$ and $L = GL_k(q) \times CSp_{2n}(q)$. We Harish-Chandra induce $\chi_{s, \lambda} \otimes \chi$ and $\chi_{s^{-1}, \lambda^{-1}} \otimes \chi$ where χ is any unipotent character of $Sp_{2n}(q)$.
- (3) We set $G = CSO_{2(n+1)}^+(q)$ and $L = GL_1(q) \times CSO_{2n}^+(q)$. We Harish-Chandra induce $\chi_{1, (1)} \otimes \chi_{s, \lambda}$ and $\chi_{1, (1)} \otimes \chi_{s, \lambda'}$.

□

8. UNIPOTENT BLOCKS OF FINITE CLASSICAL GROUPS

We continue with the assumption that p is an odd prime not dividing q . We will describe a subset of the p -blocks of our finite classical groups called the unipotent blocks. First we need the notion of linear and unitary prime [12, Section 6.1]. Let $G = GL_n(q)$, $U_n(q)$, $Sp_{2n}(q)$, $CSp_{2n}(q)$, $SO_{2n+1}(q)$, $SO_{2n}^{\pm}(q)$ or $CSO_{2n}^{\pm}(q)$.

Let d be the multiplicative order of $q \pmod p$ and p^a the maximum power of p dividing $q^d - 1$. If $G = GL_n(q)$ let $e = d$. If $G = U_n(q)$ let e be the multiplicative order of $-q_0 \pmod p$. In this case we have:

- $d = e$ if e is odd
- $d = \frac{1}{2}e$ if e is even

If G is any of the other seven groups define e to be the multiplicative order of $q^2 \pmod p$. In these cases we have:

- $e = d$ if d is odd
- $e = \frac{1}{2}d$ if d is even

When G is not a general linear group we have the notion of p being a linear or unitary prime. When we work with the unitary group p is unitary if $e = d$ and linear otherwise. For all the other groups p is linear if $e = d$ and unitary otherwise. From now on we will assume p is linear with respect to q .

8.1. Blocks. Let us now restrict our attention to $G = GL_n(q)$, $U_n(q)$, $CSp_{2n}(q)$, $SO_{2n+1}(q)$ or $CSO_{2n}^\pm(q)$. We follow [12, Section 6.2]. We will describe the unipotent blocks of $Sp_{2n}(q)$ and $SO_{2n}^\pm(q)$ at the end of this section.

First we introduce the notion of a ϕ_d -torus where ϕ_d is the d th cyclotomic polynomial. A ϕ_d -torus of an algebraic group \mathbf{G} is an F stable torus \mathbf{T} whose polynomial order is $(\phi_d)^t$ for some t . Now a d -split Levi subgroup \mathbf{K} of \mathbf{G} is a subgroup of the form $C_{\mathbf{G}}(\mathbf{T})$ for such a \mathbf{T} . A character of \mathbf{K}^F is described as d -cuspidal if it does not appear as a summand of a character Harish-Chandra induced up from any \mathbf{K}'^F where $\mathbf{K}' < \mathbf{K}$ is a d -split Levi subgroup of \mathbf{K} .

The unipotent blocks of \mathbf{G}^F are labeled by conjugacy classes (in G) of pairs (\mathbf{K}, ψ) where \mathbf{K} is a d -split Levi subgroup of \mathbf{G} and ψ is a d -cuspidal character of \mathbf{K}^F .

Now we take $G = G_m(q) = \mathbf{G}^F$ to be one of our six groups and p a linear prime with respect to q where appropriate. A typical unipotent block of G is labeled by (\mathbf{K}, ψ) where $\mathbf{K}^F \cong GL_1(q^d)^t \times G_{m'}(q)$ and $\psi = (1_{GL_1(q^d)})^t \otimes \chi$ where $m' = m - dt$ when G is the general linear group and $m - 2dt$ otherwise and χ is a unipotent character of $G_{m'}(q)$ whose label is an e -core.

An alternative way to describe the p -blocks, where p is a linear prime with respect to q , is as follows. This description can be found in more detail in [9, Theorem D] and [10, Theorem 10B, Theorem 11E]. First take a unipotent character of G . Associated to this unipotent character we have either a partition or symbol of which we take the e -core. Our unipotent block is then just labeled by this e -core μ .

We can pair our two descriptions up by letting μ in the second description be the label of χ in the first. We call this block B_μ .

Note that for $G = CSO_{2n}^+(q)$, χ labeled by a degenerate symbol and $t > 0$ then μ would be both copies of this degenerate symbol. However, the two corresponding characters of $G = CSO_{2(n-dt)}^+(q)$ are conjugate in G and hence both labels label the same block of G .

8.2. Defect Groups and Dual Defect Groups. Consider the block of G labeled by (\mathbf{K}, ψ) . Any Sylow p -subgroup of $C_{\mathbf{G}}^0([\mathbf{K}, \mathbf{K}])^F$ is a defect group for the block [12, Section 6.2].

P. Fong and B. Srinivasan describe the concept of dual defect groups. If $D \leq G$ is a defect group of (\mathbf{K}, ψ) then D^* is naturally a subgroup of G^* . For $GL_n(q)$ and $U_n(q)$ we identify G with G^* and D with D^* . For the other four groups see [10, Section 12, Section 13]. D^* is then described as a dual defect group for (\mathbf{K}, ψ) .

8.3. Characters in Unipotent Blocks. We now describe the characters in a unipotent block [9, Section 7], [10, Section 12, Section 13]. If $\Gamma \in \Phi$ (or Δ for $GL_n(q)$ or Λ for $U_n(q)$) then we set e_Γ to be the additive of $d_\Gamma \pmod{d}$. Let $G_m(q)$ be one of our groups and

let B_μ be a unipotent block of $G_m(q)$, $\chi_{t,\lambda}$ an irreducible character of $G_m(q)$ and fix a dual defect group D^* of B_μ . Then $\chi_{t,\lambda}$ lies in B_μ if and only if:

- (1) t is conjugate to x for some $x \in D^*$
- (2) The e -core of λ_{X-1} is a subset of μ
- (3) The e_Γ -core of λ_Γ is empty for all other Γ

8.4. Unipotent Blocks of $Sp_{2n}(q)$ and $SO_{2n}^\pm(q)$. Let $G = Sp_{2n}(q)$ (respectively $SO_{2n}^\pm(q)$) and $\hat{G} = CSp_{2n}(q)$ (respectively $CSO_{2n}^\pm(q)$). We now deal with the unipotent blocks of $\mathcal{O}G$ via the following lemma:

Lemma 8.4.1. *The unipotent blocks of $\mathcal{O}G$ are in one-to-one correspondence with those of $\mathcal{O}\hat{G}$. Furthermore if i and j are corresponding block idempotents then we have the following isomorphism of \mathcal{O} -algebras:*

$$\mathcal{O}Z(\hat{G})_p \otimes_{\mathcal{O}} \mathcal{O}Gi \cong \mathcal{O}\hat{G}j$$

The isomorphism is given by multiplication by j and the correspondence of characters from $\mathcal{O}\hat{G}j$ to $\mathcal{O}Z(\hat{G})_p \otimes_{\mathcal{O}} \mathcal{O}Gi$ is given by restriction.

Also if D is a defect group of $\mathcal{O}Gi$ then $Z(\hat{G})_p \times D$ is a defect group of $\mathcal{O}\hat{G}j$.

Before we prove the above we mention that we will label the block corresponding to i with the same label (symbol) as the block corresponding to j .

Proof. Apply [4, Theorem 12] to $\mathbf{G} = Sp_{2n}(\overline{\mathbb{F}}_q)$ and $CSp_{2n}(\overline{\mathbb{F}}_q)$ (respectively $SO_{2n}^\pm(\overline{\mathbb{F}}_q)$ and $CSO_{2n}^\pm(\overline{\mathbb{F}}_q)$). The lemma then follows from chasing the appropriate character correspondences and then applying [4, Proposition 6]. \square

9. MAIN THEOREM

Let $G = G_m(q)$ be one of the following groups:

- (1) $U_n(q)$
- (2) (a) $Sp_{2n}(q)$
(b) $CSp_{2n}(q)$
- (3) $SO_{2n+1}(q)$
- (4) (a) $SO_{2n}^\pm(q)$
(b) $CSO_{2n}^\pm(q)$

We continue with our assumption that q is odd in all except case (1) where it can be even or odd. As in previous sections let p an odd prime not dividing q . We also assume that p is a linear prime with respect to q . d and e will have the meaning given in 8.

For each w ($0 \leq w < p$) we wish to find an integer $m = m(w)$ and a unipotent block B_ρ of $G_m(q)$ such that theorem 1.0.2 holds. We now describe conditions for m and ρ to satisfy in each of the four cases.

- 1 ρ is an e -core partition with a representation on a $2d$ -abacus such that the i th runner has at least $w-1$ fewer beads than the $(i+2)$ th runner for $(0 \leq i \leq 2d-3)$. If r is the rank of ρ then $m = r + 2dw$.

2,3,4 There exists a $2d$ -linear diagram of ρ such that the i th runner has at least $w - 1$ fewer beads than the $(i + 1)$ th runner for $(0 \leq i \leq d - 2)$ and $(d \leq i \leq 2d - 2)$. ρ has non-zero rank r and $m = 2(r + dw) + 1$ in case 3 and $2(r + dw)$ in cases 2 and 4.

We now fix m and ρ along with a $2d$ -abacus representation of ρ such that the above property holds and every position on the first w rows is occupied with a bead.

We need to set up some notation that is required for us to state and prove the main theorem.

We set $\tilde{G} = \tilde{G}_m(q)$ to be $O_{2n+1}(q)$ (respectively $O_{2n}^\pm(q)$, $CO_{2n}^\pm(q)$) when $G = SO_{2n+1}(q)$ (respectively $SO_{2n}^\pm(q)$, $CSO_{2n}^\pm(q)$) and G otherwise. Let T_i be the subgroup of \tilde{G} generated by

$$\begin{pmatrix} I_{d(i-1)} & & & \\ & & I_d & \\ & & I_{m-2di} & \\ & -I_d & & \\ & & & I_{d(i-1)} \end{pmatrix}$$

in case 2 and by

$$\begin{pmatrix} I_{d(i-1)} & & & \\ & & I_d & \\ & & I_{m-2di} & \\ & I_d & & \\ & & & I_{d(i-1)} \end{pmatrix}$$

in cases 1,3 and 4.

Let $G = G_0 > G_1 > \cdots > G_w = L$ be a sequence of Levi subgroups of G where $G_i = L_{m,(d^i)}(q) \cong GL_d(q)_1 \times \cdots \times GL_d(q)_i \times G_{m-2di}(q)$ as described in 5.4.

We denote by H the subgroup $(GL_d(q)_1.T_1) \times \cdots \times (GL_d(q)_w.T_w) \times \tilde{G}_{m-2dw}(q)$ of $\tilde{G}_m(q)$. Note that if we adopt our notation from the introduction we have H is naturally isomorphic to $(GL_d(q).2)^w \times \tilde{G}_{m-2dw}(q)$.

Now consider the Levi subgroup $L_{m,(dw)}(q) \cong GL_{dw}(q) \times G_{m-2dw}(q) \geq L$ of G . Let S be the subgroup of permutation matrices of $GL_{dw}(q) \leq L_{m,(dw)}(q)$ whose conjugation action permutes the $GL_d(q)_i$ s. Clearly S normalises H and intersects it trivially so $H.S \cong ((GL_d(q).2) \wr S_w) \times \tilde{G}_{m-2dw}(q)$. We set $N = H.S \cap G$ and have in all cases that $|N| = 2^w w! |L|$.

For $(0 < i \leq w)$ let a_i be the principal block idempotent of $\mathcal{O}GL_d(q)_i$ and for $(0 \leq i \leq w)$ let f_{w-i} be the unipotent block idempotent of $\mathcal{O}G_{m-2di}(q)$ associated with the partition ρ . We set b_i to be the block idempotent $a_1 \otimes \cdots \otimes a_i \otimes f_{w-i}$ of $\mathcal{O}G_i$. We set $b = b_0$ and

$f = b_w$. When ρ is degenerate in case 4 we denote by f_0 and f'_0 the 2 blocks of $G_{m-2dw}(q)$ labeled by ρ and by $f = b_w$ and $f' = b'_w$ the block idempotents $a_1 \otimes \cdots \otimes a_w \otimes f_0$ and $a_1 \otimes \cdots \otimes a_w \otimes f'_0$ of $\mathcal{O}L$ respectively.

Fix a Sylow p -subgroup R of $GL_d(q)$ (note that $|R| = p^a$ see 8 for the meaning of a) and let $P_1 \times \cdots \times P_w$ be w copies of R , one in each $GL_d(q)_i$ of $GL_d(q)_1 \times \cdots \times GL_d(q)_w$. We also set Z to be the subgroup of G consisting of scalar matrices and $P = Z_p.(P_1 \times \cdots \times P_w)$. P is a defect group for $\mathcal{O}Gb$ (see 8.2) and $N_G(P) \leq N$.

Additionally for $(1 \leq i \leq w)$ let U_i be the subgroup of matrices of G_{i-1} of the form:

$$\begin{pmatrix} I_{d(i-1)} & & & & \\ & I_d & * & * & \\ & & I_{m-di} & * & \\ & & & I_d & \\ & & & & I_{d(i-1)} \end{pmatrix}$$

And set $U_i^+ = \frac{1}{|U_i|} \sum_{u \in U_i} u$.

9.1. Non-Degenerate Case. In this subsection we assume that ρ is non-degenerate. We restrict our attention to cases 1, 2(b), 3 and 4(b). We will later prove the corresponding theorem for cases 2(a) and 4(a) using 8.4.1.

Theorem 9.1.1. *$\mathcal{O}Nf$ is a block of $\mathcal{O}Nf$ and is Morita equivalent to $\mathcal{O}Gb$.*

For the the proof of this theorem, which will fill this section, we follow W. Turner[19, Section 2]. We will need a number of lemmas first.

Lemma 9.1.2.

- (1) P is defect group for $\mathcal{O}G_i b_i$ for $(0 \leq i \leq w)$.
- (2) $Br_P^G(b_i) = a'_1 \otimes \cdots \otimes a'_w \otimes f_0$ where a'_i is the principal block idempotent of $C_{GL_d(q)_i}(P_i)$ ($0 \leq i \leq w$). Also $Br_P^G(U_i^+) = 1$.
- (3) N stabilizes f and as an $\mathcal{O}(N \times L)$ -module, $\mathcal{O}Nf$ is indecomposable with vertex $\Delta(P)$. In particular, $\mathcal{O}Nf$ is a block of N .
- (4) $\mathcal{O}Gb$ and $\mathcal{O}Nf$ both have defect group P and are Brauer correspondents.

Proof.

- (1) P_i is defect group for $\mathcal{O}GL_d(q)_i a_i$ and $(P_{i+1} \times \cdots \times P_w).Z_p$ is a defect group for $\mathcal{O}G_{m-2di}(q) f_{w-i}$ (see 8.2).
- (2) $C_G(P) < G_i$ so $Br_P^G(b_i) = Br_P^{G_i}(b_i)$
 $Br_P^{G_i}(b_i) = Br_{P_1}^{GL_d(q)_1}(a_1) \otimes \cdots \otimes Br_{P_i}^{GL_d(q)_i}(a_i) \otimes Br_{(P_{i+1} \times \cdots \times P_w).Z_p}^{GL_d(q)_{i+1} \times \cdots \times GL_d(q)_w \times G_{m-2dw}(q)}(f_{w-i})$
and from [3, Theorem 3.2] we see that we must get a block of the form: $a'_1 \otimes \cdots \otimes a'_w \otimes \varepsilon$
where ε is a sum of unipotent block idempotents of $G_{m-2dw}(q)$ of defect zero.
Secondly [5, Lemma 4.5] tells us that $a'_1 \otimes \cdots \otimes a'_w \otimes f_0$ must appear as a constituent.
Finally we see that no other block idempotent of $C_G(P)$ can appear as otherwise

we would have two distinct block idempotents α and β of G_i with defect group P and $Br_P^{G_i}(\alpha)Br_P^{G_i}(\beta) \neq 0$. This is of course a contradiction as $Br_P^{G_i}$ is an algebra homomorphism.

The second part is clear.

- (3) N clearly stabilizes f in all cases except case 4(b). The only thing to check in this case is that $CO_{m-2dw}^\pm(q)$ stabilizes f_0 . This is clear however, by looking at part (2) and noting that conjugation by $N_G(P)$ commutes with Br_P^G .

By part(1), $\mathcal{O}Lf$ has vertex $\Delta(P)$. Since $C_G(P) \leq L$, the conjugate of $\Delta(P)$ by an element of $N \times L$ outside $L \times L$ is never conjugate to $\Delta(P)$ in $L \times L$. Consequently, the stabilizer of $\mathcal{O}Lf$ in $N \times L$ is exactly $L \times L$. So if $\mathcal{O}Nf = Ind_{L \times L}^{N \times L}(\mathcal{O}Lf)$ were decomposable then $Ind_{L \times L}^{N \times L}(\mathcal{O}Lf)$ would have an indecomposable summand, as a $\mathcal{O}(N \times L)$ -module, whose restriction to $L \times L$ has every summand without a vertex contained in $\Delta(P)$. Thus this indecomposable $\mathcal{O}(N \times L)$ -module does not have a vertex contained in $\Delta(P)$. This is of course a contradiction and so $\mathcal{O}Nf$ is indecomposable as a $\mathcal{O}(N \times L)$ -module. Its vertex is clearly contained in $\Delta(P)$ and its restriction to $L \times L$ has a summand with vertex $\Delta(P)$. So $\mathcal{O}Nf$ has vertex $\Delta(P)$ as a $\mathcal{O}(N \times L)$ -module.

- (4) P is a defect group for $\mathcal{O}Gb$ by part (1). Secondly we note that any p -subgroup of N is contained in L ($L \triangleleft N$ and $p \nmid [N : L]$). This tells us that $\mathcal{O}Lf$ has the same defect group as $\mathcal{O}Nf$. So we have that $\mathcal{O}Nf$ has defect group P . Finally we have that $Br_P^N(f) = Br_P^G(b)$ by (2) and so $\mathcal{O}Nf$ is the Brauer correspondent of $\mathcal{O}Gb$ in N .

□

By Alperin's description of the Brauer correspondence[1, Chapter 14, Theorem 2] the $\mathcal{O}(G \times G)$ -module $\mathcal{O}Gb$ and the $\mathcal{O}(N \times N)$ -module $\mathcal{O}Nf$ both have vertex $\Delta(P)$ and are Green correspondents. Let X be the Green correspondent of $\mathcal{O}Gb$ in $G \times N$. Then X is the unique indecomposable summand of $Res_{G \times N}^{G \times G}(\mathcal{O}Gb)$ with vertex $\Delta(P)$ and $\mathcal{O}Nf$ is the unique indecomposable summand of $Res_{N \times N}^{G \times N}(X)$ with vertex $\Delta(P)$. It is then clear that $bX = X$ and that $Xf \neq 0$, and so $Xf = X$ and X is an $(\mathcal{O}Gb, \mathcal{O}Nf)$ -bimodule.

Let Y be ${}_G Y_L = \mathcal{O}Gb_0 U_1^+ b_1 \dots U_w^+ b_w$, an $(\mathcal{O}Gb, \mathcal{O}Lf)$ -bimodule. So the functor $Y \otimes_{\mathcal{O}L} -$ from $\mathcal{O}L$ -mod to $\mathcal{O}G$ -mod is $HCInd_{G_1 b_1}^{G_0 b_0} \dots HCInd_{G_1 b_w}^{G_0 b_{w-1}}$.

Proposition 9.1.3. *There is a sequence of \mathcal{O} -split monomorphisms of algebras*

$$\mathcal{O}Nf \hookrightarrow \text{End}_{\mathcal{O}G}(X) \hookrightarrow \text{End}_{\mathcal{O}G}(Y)$$

Also the left $\mathcal{O}Gb$ -module X is a progenerator for $\mathcal{O}Gb$.

Proof. ${}_G \mathcal{O}Gb_G$ is isomorphic to a direct summand of $Ind_{G \times N}^{G \times G}({}_G X_N)$ as they are Green correspondents. Thus ${}_G \mathcal{O}Gb$ is a direct summand of $[G : N]$ copies of ${}_G X$ and ${}_G X$ is a progenerator for $\mathcal{O}Gb$.

Now there is an \mathcal{O} -split homomorphism of algebras $\mathcal{O}Nf \rightarrow \text{End}_{\mathcal{O}G}(X)$ given by multiplying on the right of X . Since ${}_N \mathcal{O}Nf_N$ is a direct summand of $Res_{N \times N}^{G \times N}({}_G X_N)$ this

homomorphism is an \mathcal{O} -split monomorphism.

Next $\text{Res}_{G \times L}^{G \times N}(GX_N)$ is indecomposable with vertex $\Delta(P)$. First, $G \times L$ contains $\Delta(P)$ so GX_N is a direct summand of $\text{Ind}_{G \times L}^{G \times N}(\text{Res}_{G \times L}^{G \times N}(GX_N))$. Since $(G \times L) \trianglelefteq (G \times N)$ there exists an indecomposable summand M of $\text{Res}_{G \times L}^{G \times N}(GX_N)$ such that $\text{Res}_{G \times L}^{G \times N}(GX_N)$ is a direct sum of conjugates of M in $G \times N$. It is possible to pick a set of coset representatives of $(G \times L)$ in $(G \times N)$ that all normalise $\Delta(P)$. So $\text{Res}_{G \times L}^{G \times N}(GX_N)$ is the sum of indecomposable modules all with vertex $\Delta(P)$.

Secondly, $\text{Res}_{G \times L}^{G \times N}(GX_N) =_G X_N \otimes_{\mathcal{O}N} \mathcal{O}Nf_L$ is a direct summand of $\text{Ind}_{N \times L}^{G \times L}(\mathcal{O}Nf)$ which by Green correspondence has exactly one summand with vertex $\Delta(P)$. So GX_L is indecomposable with vertex $\Delta(P)$.

Now we claim that GX_L is the only summand of $\text{Res}_{G \times L}^{G \times G}(G\mathcal{O}Gb_G)$ with vertex containing $\Delta(P)$. In a direct decomposition of $\text{Res}_{G \times N}^{G \times G}(G\mathcal{O}Gb_G)$ every summand is either GX_N , has a vertex strictly smaller than $\Delta(P)$ or has a vertex that is conjugate to $\Delta(P)$ in $(G \times G)$ but not in $(G \times N)$. So when we restrict down to $(G \times L)$ every summand is either GX_L , has a vertex strictly smaller than $\Delta(P)$ in size or has a vertex that is conjugate to $\Delta(P)$ in $(G \times G)$ but not in $(G \times N)$ so certainly not in $(G \times L)$. So GX_L is the only summand of $\text{Res}_{G \times L}^{G \times G}(G\mathcal{O}Gb_G)$ with vertex containing $\Delta(P)$.

Let $GY_L = \mathcal{O}Gb_0U_1^+b_1 \dots U_w^+b_w$. Each b_i and each U_i^+ is an idempotent in $(\mathcal{O}G)^L$. These idempotents commute with each other and so their product is an idempotent contained in $(\mathcal{O}G)^L$. Hence, GY_L is also a direct summand of $G\mathcal{O}Gb_L$. We show that it has as direct summands all summands of $G\mathcal{O}Gb_L$ with vertex containing $\Delta(P)$, using the Brauer homomorphism. This tells us $GY_L =_G X_L \oplus *$ and consequently we have an \mathcal{O} -split monomorphism $\text{End}_{\mathcal{O}G}(X) \hookrightarrow \text{End}_{\mathcal{O}G}(Y)$. The calculation goes:

$$\begin{aligned} Y(\Delta(P)) &= \mathcal{O}Gb_0U_1^+b_1 \dots U_w^+b_w(\Delta(P)) \\ &= kC_G((P))Br_P^G(b_0U_1^+b_1 \dots U_w^+b_w) \\ &= kC_G((P))Br_P^G(b_0)Br_P^G(U_1^+)Br_P^G(b_1) \dots Br_P^G(U_w^+)Br_P^G(b_w) \\ &= kC_G((P))Br_P^G(b_0)Br_P^G(b_1) \dots Br_P^G(b_w) \\ &= kC_G((P))Br_P^G(b) \\ &= \mathcal{O}Gb(\Delta(P)) \end{aligned}$$

So Y has indeed as summands all summands of $G\mathcal{O}Gb_L$ with vertex containing $\Delta(P)$. \square

Let φ be the character of KLf as a representation of L .

Proposition 9.1.4. *The \mathcal{O} -rank of $\mathcal{O}Nf$ and $\text{End}_{\mathcal{O}G}(Y)$ are both equal to $2^w w! \dim_K(KLf)$.*

Before we prove the above proposition we state the following lemma of Chuang and Kessar[6, Lemma 4.2] with out proof.

Lemma 9.1.5. *Let d be a positive integer and σ a partition equal to its own d -core with a d -abacus representation such that the i th runner has at least $w - 1$ fewer beads than the $(i + 1)$ th runner for $0 \leq i \leq d - 2$ and fix this abacus representation of this core. Let λ be a partition with d -core σ and weight $v \leq w$. Let μ be a partition such that $\mu_i \leq \lambda_i$ for all i with d -core σ and weight $v - 1$. Then μ is obtained by removing a d -hook from λ . If this removal occurs on the α th runner then the complement of the Young diagram of μ in that of λ is the Young diagram of the hook partition $(\alpha + 1, 1^{(d-\alpha-1)})$.*

Proof. (of 9.1.4) Since \mathcal{O} is a principal ideal domain, proving the proposition is equivalent to showing that $K \otimes_{\mathcal{O}} \mathcal{O}Nf$ and $K \otimes_{\mathcal{O}} \text{End}_{\mathcal{O}G}(Y)$ have the same dimension over K . Now $K \otimes_{\mathcal{O}} \mathcal{O}Nf \cong KNf \cong \text{Ind}_L^N(KLf)$. So $\dim_K(K \otimes_{\mathcal{O}} \mathcal{O}Nf) = 2^w w! \dim_K(KLf)$. Also $K \otimes_{\mathcal{O}} \text{End}_{\mathcal{O}G}(Y) \cong \text{End}_{KG}(K \otimes_{\mathcal{O}} Y)$. Additionally we have $K \otimes_{\mathcal{O}} Y \cong (K \otimes_{\mathcal{O}} Y) \otimes_{KLf} KLf$. Let φ be the character of KLf as a character of L . We will calculate $R_{G_1, b_1}^{G_0, b_0} \dots R_{G_w, b_w}^{G_{w-1}, b_{w-1}}(\varphi)$.

Let ρ^0 and ρ^1 be the two partitions associated to ρ as in 2.3. Our condition 9 on ρ demands that ρ^0 and ρ^1 both satisfy the conditions of σ in 9.1.5 for our d and w . We will now explain what this means in terms of Harish-Chandra induction 7:

Let $0 \leq i \leq w - 1$ and $\chi_{1, \tau}$ be a unipotent character of $G_{m-2d(i+1)}(q)$ such that τ has e -core ρ . Then:

$$R_{GL_d(q) \times G_{m-2d(i+1)}(q), a_{i+1} \otimes f_{w-i-1}}^{G_{m-2di}(q), f_{w-i}} (\chi_{1, (\alpha+1, 1^{(d-\alpha-1)})} \otimes \chi_{1, \tau})$$

is equal to the sum of the $\chi_{1, \lambda}$ s where λ is a partition obtained from τ by sliding a bead 1 place down the 2α th or $(2\alpha + 1)$ th runner in case 1 and the α th or $(\alpha + d)$ th runner in cases 2, 3 and 4.

Let us count the number of ways of sliding single beads down the l th runner of a core j times, so that on the resulting runner the bottom bead has been moved down σ_1^l times, the second bottom bead has been moved down σ_2^l times, etc, so that $\sigma_1^l \geq \sigma_2^l \geq \dots$ and $\sum_i \sigma_i^l = j$. It is equal to the number of ways of writing the numbers $1, \dots, j$ in the Young diagram of $\sigma_1^l, \sigma_2^l, \dots$ so that numbers increase across rows and down columns, that is, the degree of the character ζ^{σ^l} of the symmetric group S_j .

The irreducible characters in the block KLf are of the form $\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho}$ where either s_i is 1 and λ_i is an d -hook partition or s_i is a non-trivial p -element of $GL_d(q)$ and λ_i is the partition (1) 8.3. Note that s is always 1 or something whose image under π is 1 in cases 2(b) and 4(b). In other words, in the latter case, s is just a p -element of the underlying field in the special Clifford group.

We can now describe $R_{G_1, b_1}^{G_0, b_0} \dots R_{G_w, b_w}^{G_{w-1}, b_{w-1}}(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho})$. Suppose that

the s_i s are grouped together with like elements such that s_1, \dots, s_{r_0} are all equal to 1 and the remaining elements are grouped together into conjugacy classes as follows:

$$\begin{aligned} s_{r_0+1} &\sim \dots \sim s_{r_0+\alpha_1} \sim t_1, \\ s_{r_0+\alpha_1+1} &\sim \dots \sim s_{r_0+\alpha_1+\beta_1} \sim \overline{t_1}, \\ s_{r_0+\alpha_1+\beta_1+1} &\sim \dots \sim s_{r_0+\alpha_1+\beta_1+\alpha_2} \sim t_2, \\ s_{r_0+\alpha_1+\beta_1+\alpha_2+1} &\sim \dots \sim s_{r_0+\alpha_1+\beta_1+\alpha_2+\beta_2} \sim \overline{t_2}, \\ &\dots \end{aligned}$$

where \overline{t} means t^{-1} (or t^{-q_0} in case 1). Additionally set $r_i = \alpha_i + \beta_i$ giving $\sum_i r_i = w$ and:

$$\begin{aligned} \lambda_1 &= \dots = \lambda_{l_0} = (1^d), \\ \lambda_{l_0+1} &= \dots = \lambda_{l_0+l_1} = (2, 1^{d-2}), \\ &\dots, \\ \lambda_{l_0+\dots+l_{d-2}+1} &= \dots = \lambda_{l_0+\dots+l_{d-1}} = (d) \end{aligned}$$

where $\sum_i l_i = r_0$.

Now we have an expression for $R_{G_1, b_1}^{G_0, b_0} \dots R_{G_w, b_w}^{G_{w-1}, b_{w-1}}(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho})$:

$$\begin{aligned} \sum_{\substack{l_0+\dots+l_d=r_0 \\ |\sigma^i|+|\tau^i|=l_i, |\nu^i|=r_i}} &\binom{l_0}{|\sigma^0|} \dim \zeta^{\sigma^0} \dim \zeta^{\tau^0} \dots \binom{l_{d-1}}{|\sigma^{d-1}|} \dim \zeta^{\sigma^{d-1}} \dim \zeta^{\tau^{d-1}} \\ &\dim \zeta^{\nu^1} \dim \zeta^{\nu^2} \dots \\ &\chi(s \times s_1 \times \overline{s_1} \times \dots \times s_w \times \overline{s_w}, \mu) \end{aligned}$$

Where μ_{X-1} is the partition/symbol whose e -core is ρ and whose e -quotients are $[\sigma^0, \dots, \sigma^{d-1}]$ and $[\tau^0, \dots, \tau^{d-1}]$ (or just $[\sigma^0, \tau^0, \dots, \sigma^{d-1}, \tau^{d-1}]$ in case 1) with respect to the $2d$ -abacus representation of ρ already fixed 9 and $\mu_{\psi_i} = \nu^i$ where ψ_i is the minimal polynomial of $t_i \times \overline{t_i}$.

Now we can permute the λ_i s and still get the same character of G when we do Harish-Chandra induction. There are exactly $w!/(l_0!l_1!\dots l_{d-1}!\alpha_1!\beta_1!\alpha_2!\beta_2!\dots)$ such permutations. Also each irreducible character appears in φ with multiplicity equal to the dimension of said character. So we have the following expression for $R_{G_1, b_1}^{G_0, b_0} \dots R_{G_w, b_w}^{G_{w-1}, b_{w-1}}(\varphi)$:

$$\begin{aligned}
& \sum_{\substack{l_0+\dots+l_{d-1}+r_1+r_2+\dots=w \\ \alpha_i+\beta_i=r_i, |\sigma^i|+|\tau^i|=l_i, \kappa}} \frac{w!}{l_0!l_1!\dots l_{d-1}!\alpha_1!\beta_1!\alpha_2!\beta_2!\dots} \\
& \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho}) \\
& \binom{l_0}{|\sigma^0|} \dim \zeta^{\sigma^0} \dim \zeta^{\tau^0} \dots \binom{l_{d-1}}{|\sigma^{d-1}|} \dim \zeta^{\sigma^{d-1}} \dim \zeta^{\tau^{d-1}} \\
& \dim \zeta^{\nu^1} \dim \zeta^{\nu^2} \dots \\
& \chi(s \times s_1 \times \overline{s_1} \times \dots \times s_w \times \overline{s_w}, \mu)
\end{aligned}$$

Where s is the scalar matrix κI . So κ runs over $(\mathbb{F}_q^\times)_p$ in cases 2(b) and 4(b) and just 1 otherwise. So we get:

$$\begin{aligned}
& \sum_{\substack{l_0+\dots+l_{d-1}+r_1+r_2+\dots=w \\ |\sigma^i|+|\tau^i|=l_i, \kappa}} \frac{w!}{l_0!l_1!\dots l_{d-1}!r_1!r_2!\dots} \\
& \binom{l_0}{|\sigma^0|} \dim \zeta^{\sigma^0} \dim \zeta^{\tau^0} \dots \binom{l_{d-1}}{|\sigma^{d-1}|} \dim \zeta^{\sigma^{d-1}} \dim \zeta^{\tau^{d-1}} \\
& \dim \zeta^{\nu^1} \dim \zeta^{\nu^2} \dots \\
& \chi(s \times s_1 \times \overline{s_1} \times \dots \times s_w \times \overline{s_w}, \mu) \\
& \left[\sum_{\alpha_i+\beta_i=r_i} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} \dots \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho}) \right]
\end{aligned}$$

So the dimension of $\text{End}_{KG}((K \otimes_{\mathcal{O}} Y) \otimes_K KLf)$ over K is:

$$\begin{aligned}
& \sum_{\substack{l_0+\dots+r_1+\dots=w \\ |\sigma^i|+|\tau^i|=l_i, \kappa}} \left(\frac{w!}{l_0!l_1!\dots l_{d-1}!r_1!r_2!\dots} \right)^2 \\
& \binom{l_0}{|\sigma^0|}^2 (\dim \zeta^{\sigma^0})^2 (\dim \zeta^{\tau^0})^2 \dots \binom{l_{d-1}}{|\sigma^{d-1}|}^2 (\dim \zeta^{\sigma^{d-1}})^2 (\dim \zeta^{\tau^{d-1}})^2 \\
& (\dim \zeta^{\nu^1})^2 (\dim \zeta^{\nu^2})^2 \dots \\
& \left[\sum_{\alpha_i+\beta_i=r_i} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} \dots \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho}) \right]^2
\end{aligned}$$

$$\begin{aligned}
= & \sum_{\substack{l_0+\dots+r_1+\dots=w \\ \sigma_i+\tau_i=l_i, \kappa}} \left(\frac{w!}{l_0!l_1!\dots l_{d-1}!r_1!r_2!\dots} \right)^2 \binom{l_0}{\sigma_0}^2 \dots \binom{l_{d-1}}{\sigma_{d-1}}^2 \\
& \left[\sum_{\sigma^i \vdash \sigma_i, \tau^i \vdash \tau_i} (\dim \zeta^{\sigma^0})^2 (\dim \zeta^{\tau^0})^2 \dots (\dim \zeta^{\sigma^{d-1}})^2 (\dim \zeta^{\tau^{d-1}})^2 \right] \\
& (\dim \zeta^{\nu^1})^2 (\dim \zeta^{\nu^2})^2 \dots \\
& \left[\sum_{\alpha_i+\beta_i=r_i} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} \dots \dim(\chi_{s_1, \lambda_1}) \dots \dim(\chi_{s_w, \lambda_w}) \dim(\chi_{s, \rho}) \right]^2
\end{aligned}$$

Now using the fact that $\sum_{\sigma \vdash h} \dim(\zeta^\sigma)^2 = h!$ we get:

$$\begin{aligned}
& \sum_{\substack{l_0+\dots+r_1+\dots=w \\ \sigma_i+\tau_i=l_i, \kappa}} \left(\frac{w!}{l_0!l_1!\dots l_{d-1}!r_1!r_2!\dots} \right)^2 \binom{l_0}{\sigma_0}^2 \dots \binom{l_{d-1}}{\sigma_{d-1}}^2 \\
& \sigma_0!\tau_0!\dots \sigma_{d-1}!\tau_{d-1}!r_1!r_2!\dots \\
& \left[\sum_{\alpha_i+\beta_i=r_i} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} \dots \dim(\chi_{s_1, \lambda_1}) \dots \dim(\chi_{s_w, \lambda_w}) \dim(\chi_{s, \rho}) \right]^2
\end{aligned}$$

Now $\dim(\chi_{t, (1)}) = \dim(\chi_{\bar{t}, (1)})$ 7.3. This means that for fixed (r_1, r_2, \dots) the choice of the α_i s does not affect $\dim(\chi_{s_1, \lambda_1}) \dots \dim(\chi_{s_w, \lambda_w}) \dim(\chi_{s, \rho})$. Using this and the fact that $\sum_{i=0}^r \binom{r}{i} = 2^r$ we get:

$$\begin{aligned}
& w! \sum_{\substack{l_0+\dots+r_1+\dots=w \\ \sigma_i+\tau_i=l_i, \kappa}} \frac{w!}{l_0!l_1!\dots l_{d-1}!r_1!r_2!\dots} \binom{l_0}{\sigma_0} \dots \binom{l_{d-1}}{\sigma_{d-1}} \\
& 2^{\sum r_i} \left[\sum_{\alpha_i+\beta_i=r_i} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} \dots \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho})^2 \right]
\end{aligned}$$

$$\begin{aligned}
= & w! \sum_{\substack{l_0+\dots+r_1+\dots=w \\ \kappa}} \frac{w!}{l_0!l_1!\dots l_{d-1}!r_1!r_2!\dots} \\
& 2^{\sum r_i} 2^{\sum l_i} \left(\sum_{\alpha_i+\beta_i=r_i} \binom{r_1}{\alpha_1} \dots \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho})^2 \right)
\end{aligned}$$

$$\begin{aligned}
&= 2^w w! \sum_{\substack{l_0 + \dots + l_{d-1} + \alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \dots = w \\ \kappa}} \frac{w!}{l_0! l_1! \dots l_{d-1}! \alpha_1! \beta_1! \alpha_2! \beta_2! \dots} \\
&\quad \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho})^2 \\
&= 2^w w! \dim_K(KL f)
\end{aligned}$$

□

Proof. (of 9.1.1) By 9.1.4 $\mathcal{O}Nf$ and $\text{End}_{\mathcal{O}G}(Y)$ have the same \mathcal{O} -rank. As a consequence we have that all the monomorphisms in 9.1.3 become isomorphisms. Therefore, since X is a progenerator as a left $\mathcal{O}Gb$ -module, ${}_{\mathcal{O}Gb}X_{\mathcal{O}Nf}$ induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Nf$. □

9.2. Degenerate Case. We now assume that ρ is degenerate and restrict our attention to case 4(b). Again we will later prove the corresponding theorem for case 4(a) using 8.4.1.

Theorem 9.2.1. $\mathcal{O}N(f + f')$ is a block of N and is Morita equivalent to $\mathcal{O}Gb$.

The proof of this theorem will closely resemble that of 8.4.1. We will need all the corresponding lemmas first.

Lemma 9.2.2.

- (1) P is defect group for $\mathcal{O}G_i b_i$ for $(0 \leq i \leq w-1)$ and also for $\mathcal{O}Lf$ and $\mathcal{O}Lf'$.
- (2) $Br_P^G(b_i) = Br_P^G(f) = Br_P^G(f') = a'_1 \otimes \dots \otimes a'_w \otimes (f_0 + f'_0)$ where a'_i is the principal block of $C_{GL_d(q)_i}(P_i)$ for $(0 \leq i \leq w-1)$. In addition we have $Br_P^G(U_i^+) = 1$.
- (3) N stabilizes $(f + f')$, $\mathcal{O}Nf$ and $\mathcal{O}Nf'$ are both indecomposable with vertex $\Delta(P)$ and $\mathcal{O}N(f + f')$ is a block of N .
- (4) $\mathcal{O}Gb$ and $\mathcal{O}N(f + f')$ both have defect group P and are Brauer correspondents.

Proof.

- (1) P_i is defect group for $\mathcal{O}GL_d(q)_i a_i$ and $(P_{i+1} \times \dots \times P_w).Z_p$ is a defect group for $\mathcal{O}G_{m-di}(q)f_{w-i}$. Similarly for $\mathcal{O}Lf$ and $\mathcal{O}Lf'$.
- (2) Identical to 9.1.2 part (2).
- (3) As with the non-degenerate case the only thing to check for the first part is that $CO_{m-2dw}^+(q)$ stabilizes $(f + f')$. This is again clear as in the non-degenerate case. Also as in the non-degenerate case we have both $\mathcal{O}Nf$ and $\mathcal{O}Nf'$ are indecomposable as $\mathcal{O}(N \times L)$ -modules with vertex $\Delta(P)$. Note that $\mathcal{O}Nf$ and $\mathcal{O}Nf'$ lie in different blocks of $\mathcal{O}(N \times L)$. This implies that if $\mathcal{O}N(f + f')$ were decomposable as a $\mathcal{O}(N \times N)$ -module we would have to get the same decomposition $\mathcal{O}N(f + f') \cong \mathcal{O}Nf \oplus \mathcal{O}Nf'$. However $N_G(P) \leq N$ is transitive on the blocks of $C_G(P)$ appearing in the image of b under Br_P^G . This means f_0 is conjugate to f'_0 and hence f to f' . Thus $\mathcal{O}N(f + f')$ is one block of N .

- (4) P is a defect group for $\mathcal{O}Gb$ by part (1). As in the non-degenerate case every p -subgroup of N lies in L so $\mathcal{O}N(f + f')$ has defect group P . Also $Br_P^N(f + f') = Br_P^G(b)$ by (2) and so $\mathcal{O}N(f + f')$ is the Brauer correspondent of $\mathcal{O}Gb$ in N . \square

We again let X be the Green correspondent of $\mathcal{O}Gb$ in $G \times N$. This time we set $Y = {}_G Y_L = \mathcal{O}Gb_0 U_1^+ b_1 \dots U_w^+(b_w + b'_w)$. Recall that $f = b_w$ and $f' = b'_w$.

Proposition 9.2.3. *There is a sequence of \mathcal{O} -split monomorphisms of algebras*

$$\mathcal{O}N(f + f') \hookrightarrow \text{End}_{\mathcal{O}G}(X) \hookrightarrow \text{End}_{\mathcal{O}G}(Y)$$

Also the left $\mathcal{O}Gb$ -module X is a progenerator for $\mathcal{O}Gb$.

During the proof we will refer back to the corresponding proposition for the non-degenerate case 9.1.3.

Proof. As in the non-degenerate case X is clearly a progenerator for $\mathcal{O}Gb$. It is also clear that we have the \mathcal{O} -split monomorphism $\mathcal{O}N(f + f') \hookrightarrow \text{End}_{\mathcal{O}G}(X)$.

This time ${}_G X_L$ is not indecomposable. Instead we have that it is the direct sum of 2 indecomposable modules Xf and Xf' each with vertex $\Delta(P)$. Next, as with the non-degenerate case, we have that ${}_G X_L$ is the direct sum of indecomposable modules all with vertex $\Delta(P)$. Finally we see that ${}_G X_L f = {}_G X_N \otimes_{\mathcal{O}N} \mathcal{O}N f_L$ is a direct summand of $\text{Ind}_{N \times L}^{G \times L}(\mathcal{O}N f)$ which by Green correspondence has exactly one summand with vertex $\Delta(P)$. So ${}_G X_L f$ is indecomposable with vertex $\Delta(P)$. Similarly for ${}_G X_L f'$.

The proof that we have an \mathcal{O} -split monomorphism $\text{End}_{\mathcal{O}G}(X) \hookrightarrow \text{End}_{\mathcal{O}G}(Y)$ is again as in the non-degenerate case with b_w replaced by $b_w + b'_w$. \square

Let φ be the character of $KL(f + f')$ as a representation of L .

Proposition 9.2.4. *The \mathcal{O} -rank of $\mathcal{O}N(f + f')$ and $\text{End}_{\mathcal{O}G}(Y)$ are both equal to $2^w w! \dim_K(KL f)$.*

Proof. We obtain a combinatorial rule for Harish-Chandra induction in this case by comparing with the non-degenerate case where everything is calculated in the Weyl group of type B . We first want to calculate

$$R_{G_{w-v+1}, b_{w-v+1}}^{G_{w-v}, b_{w-v}} \dots R_{G_{w, (b_w + b'_w)}}^{G_{w-1}, b_{w-1}} (\chi_{1, \lambda_1} \otimes \dots \otimes \chi_{1, \lambda_v} \otimes (\chi_{s, \rho} + \chi'_{s, \rho}))$$

where $v \leq w$ and s is a p -element of the underlying field \mathbb{F}_q .

Let $\tau = \{X, Y\}$ be a non-degenerate symbol of weight $w - v$. Then the multiplicity of $\chi_{s, \tau}$ is just the number of ways of obtaining τ from ρ by sliding beads down the appropriate runners. (Appropriate means the runners determined by the λ_i s. Also obtaining τ from ρ means obtaining τ so that X is on the left or the right.)

Now let $\tau = \{X, X\}$ be a degenerate symbol of weight $w - v$. Then again the multiplicity of $\chi_{s,\tau}$ is just the number of ways of obtaining τ from ρ by sliding beads down the appropriate runners. The same is true of $\chi'_{s,\tau}$.

Adopting the same notation as in the non-degenerate case we let φ be the character of $KL(f + f')$. We have the following expression for $R_{G_1, b_1}^{G_0, b_0} \dots R_{G_w, (b_w + b'_w)}^{G_{w-1}, b_{w-1}}(\varphi)$:

$$\sum_{\substack{l_0 + \dots + l_{d-1} + r_1 + r_2 + \dots = w \\ \alpha_i + \beta_i = r_i, |\sigma^i| + |\tau^i| = l_i, \kappa}} \frac{w!}{l_0! l_1! \dots l_{d-1}! \alpha_1! \beta_1! \alpha_2! \beta_2! \dots} \\ \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho}) \\ \binom{l_0}{|\sigma^0|} \dim \zeta^{\sigma^0} \dim \zeta^{\tau^0} \dots \binom{l_{d-1}}{|\sigma^{d-1}|} \dim \zeta^{\sigma^{d-1}} \dim \zeta^{\tau^{d-1}} \\ \dim \zeta^{\nu^1} \dim \zeta^{\nu^2} \dots \\ \chi(s \times s_1 \times \overline{s_1} \times \dots \times s_w \times \overline{s_w}, \mu)$$

Compare with the non-degenerate case. $\chi(s \times s_1 \times \overline{s_1} \times \dots \times s_w \times \overline{s_w}, \mu)$ means the sum of both characters when μ_{X-1} is degenerate. Note that $\dim(\chi_{s, \rho}) = \dim(\chi'_{s, \rho})$ (see 7.3).

$$= \sum_{\substack{l_0 + \dots + l_{d-1} + r_1 + r_2 + \dots = w \\ |\sigma^i| + |\tau^i| = l_i, \kappa}} \frac{w!}{l_0! l_1! \dots l_{d-1}! r_1! r_2! \dots} \\ \binom{l_0}{|\sigma^0|} \dim \zeta^{\sigma^0} \dim \zeta^{\tau^0} \dots \binom{l_{d-1}}{|\sigma^{d-1}|} \dim \zeta^{\sigma^{d-1}} \dim \zeta^{\tau^{d-1}} \\ \dim \zeta^{\nu^1} \dim \zeta^{\nu^2} \dots \\ \chi(s \times s_1 \times \overline{s_1} \times \dots \times s_w \times \overline{s_w}, \mu) \\ \left[\sum_{\alpha_i + \beta_i = r_i} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} \dots \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho}) \right]$$

Note that if we swap all the σ^i s and τ^i s over then we get the same character. Recalling also that $\chi(s \times s_1 \times \overline{s_1} \times \dots \times s_w \times \overline{s_w}, \mu)$ means the sum of both characters when μ_{X-1} is degenerate the dimension of $\text{End}_{KG}((K \otimes_{\mathcal{O}} Y) \otimes_K KLf)$ over K is:

$$\begin{aligned}
& 2 \sum_{\substack{l_0+\dots+r_1+\dots=w \\ |\sigma^i|+|\tau^i|=l_i, \kappa}} \left(\frac{w!}{l_0!l_1!\dots l_{d-1}!r_1!r_2!\dots} \right)^2 \\
& \quad \left(\frac{l_0}{|\sigma^0|} \right)^2 (\dim \zeta^{\sigma^0})^2 (\dim \zeta^{\tau^0})^2 \dots \left(\frac{l_{d-1}}{|\sigma^{d-1}|} \right)^2 (\dim \zeta^{\sigma^{d-1}})^2 (\dim \zeta^{\tau^{d-1}})^2 \\
& \quad (\dim \zeta^{\nu^1})^2 (\dim \zeta^{\nu^2})^2 \dots \\
& \quad \left[\sum_{\alpha_i+\beta_i=r_i} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} \dots \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho}) \right]^2
\end{aligned}$$

Following all the step through as before we get:

$$\begin{aligned}
& 2^{w+1} w! \sum_{\substack{l_0+\dots+\alpha_1+\beta_1+\alpha_2+\beta_2+\dots=w \\ \kappa}} \frac{w!}{l_0!l_1!\dots l_{d-1}!\alpha_1!\beta_1!\alpha_2!\beta_2!\dots} \\
& \quad \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho})^2
\end{aligned}$$

Recall that $\dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi_{s, \rho}) = \dim(\chi_{s_1, \lambda_1} \otimes \dots \otimes \chi_{s_w, \lambda_w} \otimes \chi'_{s, \rho})$ to obtain:

$$= 2^w w! \dim_K(KLf)$$

□

Proof. (of 9.2.1) Again 9.2.4 and 9.2.3 give us the relevant information to obtain that $\mathcal{O}_{Gb}X_{\mathcal{O}N(f+f')}$ induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Nf$. □

9.3. $Sp_{2n}(q)$ and $SO_{2n}^\pm(q)$. We are now in a position to prove 9.1.1 and 9.2.1 our main theorem for cases 2(a) and 4(a).

Corollary 9.3.1.

- (1) If ρ is non-degenerate then $\mathcal{O}Nf$ is a block of N and is Morita equivalent to $\mathcal{O}Gb$.
- (2) If ρ is degenerate then $\mathcal{O}N(f+f')$ is a block of N and is Morita equivalent to $\mathcal{O}Gb$.

Of course part (2) only happens in case 4(a).

We will use the proof of the corresponding theorems for cases 2(b) and 4(b) (see 9.1.1 and 9.2.1) as well as 8.4.1.

Proof. Adopting the notation of 8.4.1 we have the correspondence between characters of $\mathcal{O}\hat{G}j$ and $\mathcal{O}Gi$ given by restriction but with each character of $\mathcal{O}Gi$ appearing as a restriction of $(q-1)_p$ distinct characters of $\mathcal{O}\hat{G}j$.

Given that this correspondence commutes with Harish-Chandra induction we have that

the proofs of 9.1.1 and 9.2.1 run through for G . Note that the character sums have to be divided by $(q-1)_p$. \square

All that remains to prove theorem 1.0.2 is the following lemma. We continue with the notation from this section as well as that from 1.0.2. We will consider all 4 cases.

Lemma 9.3.2.

- (1) If ρ is non-degenerate then $\mathcal{O}Nf$ is Morita equivalent to the principal block of $\mathcal{O}((GL_d(q).2 \wr S_w) \times Z_p)$.
- (2) If ρ is degenerate then $\mathcal{O}N(f+f')$ is Morita equivalent to the principal block of $\mathcal{O}(M \times Z_p)$.

Proof. We prove for cases 1,2(a),3 and 4(a). The statements for cases 2(b) and 4(b) will then follow from the corresponding statements for cases 2(a) and 4(a) and 8.4.1.

- (1) This is clear for cases 1 and 2(a). For cases 3 and 4(a) we take the definition of \tilde{G} from the beginning of the section. First consider the unique character of $\mathcal{O}G_{m-2dw}(q)f_0$. Since $\mathcal{O}Nf$ is a block this character is invariant under conjugation by $\tilde{G}_{m-2dw}(q)$ as $\mathcal{O}G_{m-2dw}(q)f_0$ is. Therefore this character induces to 2 different characters of $\tilde{G}_{m-2dw}(q)$ and hence $\mathcal{O}\tilde{G}_{m-2dw}(q)f_0$ is not a block and is in fact the direct sum of 2 blocks, $\mathcal{O}\tilde{G}_{m-2dw}(q)f'_0$ and $\mathcal{O}\tilde{G}_{m-2dw}(q)f''_0$ both Morita equivalent to $\mathcal{O}G_{m-2dw}(q)f_0$ due to [4, Proposition 6].

$\mathcal{O}Nf$ is Morita equivalent to the block of $\mathcal{O}((GL_d(q).2 \wr S_w) \times \tilde{G}_{m-2dw}(q))$ with block idempotent $a_1 \otimes \cdots \otimes a_w \otimes f'_0$ by [4, Proposition 6] which is in turn clearly Morita equivalent to the principal block of $\mathcal{O}(GL_d(q).2 \wr S_w)$.

- (2) Let a' be the principal block idempotent of $\mathcal{O}M$. Then $\mathcal{O}Ma'$ is Morita equivalent to $\mathcal{O}Ma' \otimes_{\mathcal{O}} \mathcal{O}G_{m-2dw}f_0$ which is in turn Morita equivalent to $\mathcal{O}N(f+f')$. \square

10. APPLICATION TO BROUÉ'S CONJECTURE

We are now in a position to show that $\mathcal{O}Gb$ is derived equivalent to its Brauer correspondent in $N_G(P)$. We will use h to denote the corresponding block idempotent of $\mathcal{O}N_G(P)$.

Corollary 10.0.3. *$\mathcal{O}Gb$ is derived equivalent to its Brauer correspondent in $N_G(P)$.*

We will need a couple of lemmas to prove the above corollary but first we define M' analogously to how we defined M in the introduction.

We have a natural homomorphism $N_{GL_d(q).2}(R) \rightarrow \{\pm 1\}$ with kernel $N_{GL_d(q)}(R)$. This extends to a map $N_{GL_d(q).2}(R) \wr S_w \rightarrow \{\pm 1\}$ and we use M' to denote the kernel of this map.

Lemma 10.0.4.

- (1) If ρ is non-degenerate then $\mathcal{O}N_G(P)h$ is Morita equivalent to the principal block of $\mathcal{O}((N_{GL_d(q).2}(R) \wr S_w) \times Z_p)$.

- (2) If ρ is degenerate then $\mathcal{O}N_G(P)h$ is Morita equivalent to the principal block of $\mathcal{O}(M' \times Z_p)$.

Proof. See proof of 9.3.2. □

We will need the following theorem due to A. Marcus[17, Theorem 3.2(b)].

Theorem 10.0.5. *Let X, \tilde{X}, Y and \tilde{Y} be finite groups with $X \triangleleft \tilde{X}$, $Y \triangleleft \tilde{Y}$, $X \leq Y$ and $\tilde{X} \leq \tilde{Y}$. We also require that $\tilde{X} \cap Y = X$ and that the natural homomorphism $\tilde{X}/X \rightarrow \tilde{Y}/Y$ is in fact an isomorphism of p' -groups. Now let i (respectively j) be a block idempotent of $\mathcal{O}X$ (respectively $\mathcal{O}Y$) which is fixed by conjugation by \tilde{X} (respectively \tilde{Y}) and $(\mathcal{C}, \mathcal{C}^*)$ a pair of complexes giving a derived equivalence between $\mathcal{O}Xi$ and $\mathcal{O}Yj$. Now consider A the subalgebra of $\mathcal{O}\tilde{X}i \otimes_{\mathcal{O}} (\mathcal{O}\tilde{Y}j)^{op}$ generated by $\mathcal{O}Xi \otimes_{\mathcal{O}} (\mathcal{O}Yj)^{op}$ and $xi \otimes x^{-1}j$ for all $x \in \tilde{X}$. Suppose that \mathcal{C} extends to complex of A -modules, then $(\mathcal{O}\tilde{X}i \otimes_{\mathcal{O}Xi} \mathcal{C}, \mathcal{O}\tilde{Y}j \otimes_{\mathcal{O}Yi} \mathcal{C}^*)$ is a pair of complexes giving a derived equivalence between $\mathcal{O}\tilde{X}i$ and $\mathcal{O}\tilde{Y}j$.*

If \mathcal{C} extends to complex of A -modules we will say that \mathcal{C} has a consistent diagonal action of \tilde{X} .

We are now in a position to prove corollary 10.0.3.

Proof. 10.0.3 We already have that $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}Nf$ 9.1.1 (or $\mathcal{O}N(f + f')$ in the degenerate case 9.2.1) which is in turn Morita equivalent to the principal block of $\mathcal{O}((GL_d(q).2 \wr S_w) \times Z_p)$ (respectively $\mathcal{O}(M \times Z_p)$) 9.3.2. On the other hand $\mathcal{O}N_G(P)h$ is Morita equivalent to the principal block of $\mathcal{O}((N_{GL_d(q).2}(R) \wr S_w) \times Z_p)$ (respectively $\mathcal{O}(M' \times Z_p)$) 10.0.4. Therefore all that remains to prove 10.0.3 is that the principal blocks of $\mathcal{O}(GL_d(q).2 \wr S_w)$ and $\mathcal{O}(N_{GL_d(q).2}(R) \wr S_w)$ (respectively $\mathcal{O}M$ and $\mathcal{O}M'$) are derived equivalent.

R is cyclic so the principal block of $\mathcal{O}GL_d(q).2$ is derived equivalent to the principal block of $\mathcal{O}N_{GL_d(q).2}(R)$. Then by [17, Theorem 4.3(b)] the principal block of $\mathcal{O}(GL_d(q).2 \wr S_w)$ is derived equivalent to the principal block of $\mathcal{O}(N_{GL_d(q).2}(R) \wr S_w)$.

By [17, Example 5.5] there exists a pair of complexes $(\mathcal{C}, \mathcal{C}^*)$ that induce a derived equivalence between the principal blocks of $\mathcal{O}N_{GL_d(q)}(R)$ and $\mathcal{O}GL_d(q)$ such that \mathcal{C} has a consistent diagonal action of $N_{GL_d(q).2}(R)$.

By [17, Theorem 4.3(b)] we can construct a derived equivalence between the principal blocks of $\mathcal{O}(N_{GL_d(q)}(R) \wr S_w)$ and $\mathcal{O}(GL_d(q) \wr S_w)$. Let $(\mathcal{D}, \mathcal{D}^*)$ be the pair of complexes giving this equivalence. Note that from the construction of $(\mathcal{D}, \mathcal{D}^*)$ and the previous paragraph that \mathcal{D} has a consistent diagonal action of $(N_{GL_d(q).2}(R)) \wr S_w$ so certainly of M' . Therefore by 10.0.5 we can construct a derived equivalence between the principal blocks of $\mathcal{O}M$ and $\mathcal{O}M'$. □

Note that lemma 10.0.4 still holds without our condition 9. Therefore we have the following corollary for all four cases.

Corollary 10.0.6. *Let B be a unipotent block of $\mathcal{O}G_{m'}(q)$ of weight w with abelian defect group P' . If B' is the Brauer correspondent of B in $N_{G_{m'}(q)}(P')$ then the Morita equivalence class of B' depends only on w and whether B is degenerate or not.*

The above corollary is an analogy of part (3) of the proof of Broué's conjecture for the symmetric group given in the introduction.

Remark 10.0.7. *We note that a corresponding theorem to 1.0.2 for the case of q being even has yet to be proven for all but the unitary group.*

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